~~Claim 2: T' is connected~~

~~Let $x \neq y \in V$~~

~~Case 1: $\exists$ an x-y path that does not use e ✓~~

~~Case 2: All x-y paths use edge e~~

~~If a path uses $e = (u, w)$, take the "scenic path" from u to w in C ✓~~

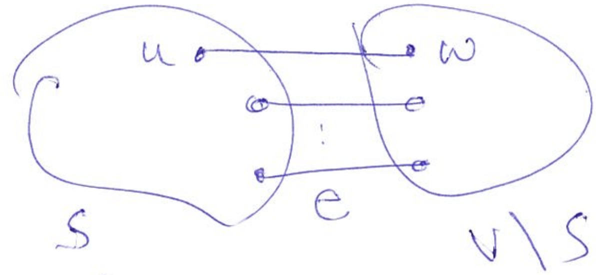
~~$\Rightarrow$ (x, y) still connected in T' $\blacktriangle$~~

Oct 25

Assume: All c_e's are distinct (we'll later remove this.)

CUT PROPERTY Lemma

For all cuts $(S, V \setminus S)$ s.t $S \neq \emptyset$ and $V \setminus S \neq \emptyset$ $\equiv$ $S \neq V$



Consider all crossing edges $(u, w) \in E$ s.t $u \in S$, $w \notin S$

Let e be the crossing edge with minimum cost.

$\Rightarrow$ e is in ALL MSTs for G.

Proof of correctness (idea): Prim/ Kruskal

every edge added by the algo is cheapest crossing edge for some cut.

Assume cut property Lemma (CPL) {+ all c_e's are distinct}

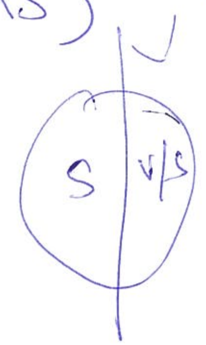
THM 1: Prim's algo is correct (i.e. for all inputs it outputs an MST)

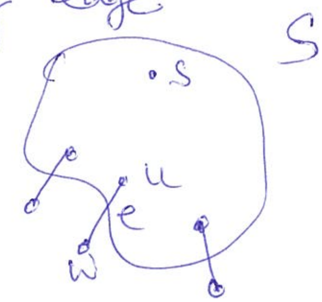
Pf (idea) Consider the state of Prim's algo when it is about to add an edge $e = (u, w)$ $\{u \in S, w \notin S\}$

in Prim's algo

Goal: Show the e is the cheapest crossing edge for some cut $(S, V \setminus S)$

→ Pick S as in Prim's algo





Claim 1: $S \neq \emptyset$ ($s \in S$, $u \in S$)

Claim 2: $S \neq V$ ($w \notin S$)

Claim 3: e is the cheapest crossing edge for $(S, V \setminus S)$ (by definition of Prim's algo)

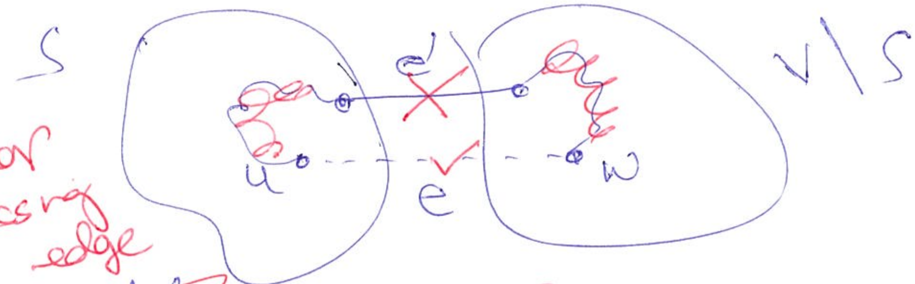
$\Rightarrow$ all edges added by Prim's are "safe" by CPL

Claim 4: At the end of each iteration (S, T) is connected ($\Rightarrow$ at the end (V, T) is connected)

Claims 1+2+3+4 + CPL $\Rightarrow$ THM 1 ∎

Pf (idea) of CPL: By contradiction.

Assume $\exists$ a cut $(S, V\setminus S)$ & an MST $T = (V, E')$ s.t. the cheapest crossing edge for $(S, V\setminus S)$ is NOT in T.



Let e' be some other crossing edge

~~Since T is connected $\Rightarrow$ $\exists$ an u-w path~~
~~$u \in S$, $w \notin S$ the path crosses the cut $(S, V\setminus S)$ at some edge e'~~

contradict

Goal: Construct a spanning tree T' s.t. $c(T') < c(T)$

$$T' = (V, (E' \setminus \{e'\}) \cup \{e\})$$

$$c(T') = c(T) - c_{e'} + c_e$$

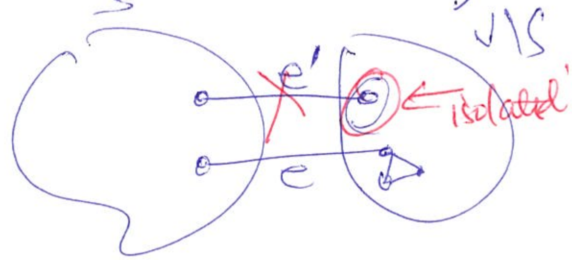
Obs: $c_{e'} > c_e$ (e is the cheapest crossing edge + all edge costs are distinct)

Obs $< c(T)$

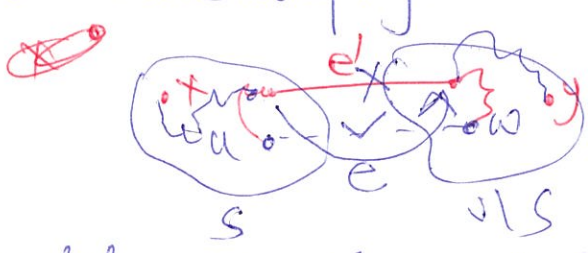
$\Rightarrow$ contradicts our claim that T is an MST ∎

Q: Where is the above "proof" wrong?

Issue: T' need not be connected



Fix! Pick e' more carefully.



T is connected $\Rightarrow$ $\exists$ u-w path in T
$u \in S$, $w \notin S$ the path has a crossing edge e'
in it

$T' = (V, (E' \setminus \{e'\}) \cup \{e\})$

Claim 1: $c(T') < c(T)$ as before

Claim 2: T' is connected

$x, y \in V$

Case 1 ~~(Claim)~~: $\exists$ an x-y path that does not use e' ✓

Case 2: $\exists$ an x-y path the does use e'

In T' use the 'scenic path'

$\Rightarrow$ x, y are still connected in T' ∎

THM 2: Kruskal's algo is correct (i.e. always outputs an MST)

→ Consider the time when Kruskal's ~~algo~~ algo is about to add the edge e

Goal: Show e is the cheapest crossing edge for some cut $(S, V \setminus S)$

$e = (u, w)$

Def: S be the set of all vertices connected to u only using edges in T