Nov 13

<u>Simplified problem</u>: Only want to compute the <u>value</u> of an optimal solution.

<u>Assume</u>: $f_1 \le f_2 \le \cdots \le f_n$

<u>Def</u>: $OPT(j)$ = value of an optimal solution for instance $[j]$ $\forall j \in [n]$

$\underbrace{\text{for instance } [j]}_{(s_1, f_1, v_1), \ldots, (s_j, f_j, v_j)}$

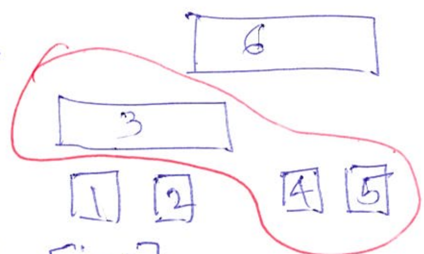
<u>Goal</u>: Compute $OPT(n)$

<u>Assume</u>: $OPT(0) = 0$

<u>Def</u>: $\forall j \in [n]$, let O_j be an optimal solution for $[j]$

$$\Rightarrow \quad OPT(j) \underset{(*)}{=} v(O_j) = \sum_{i \in O_j} v_i$$

<u>Recursive algo</u>



<u>Case 1</u>: $\boxed{j \notin O_j}$

<u>Claim 1</u>: O_j is also optimal for $[j-1]$

$$\Rightarrow OPT(j) = v(O_j) \underset{\text{Claim 1}}{=} OPT(j-1) \Rightarrow \boxed{OPT(j) = OPT(j-1)}$$

<u>Pf (idea) of Claim 1</u>: By contradiction.

Assume $\exists\ O' \subseteq [j-1] \subseteq [j]$ s.t. $v(O') > v(O_j)$

<u>Note</u>: O' is a valid schedule for $[j]$

$\Rightarrow O'$ is a valid schedule for $[j]$ AND $v(O') > v(O_j)$

$\Rightarrow O_j$ is <u>not</u> optimal for $[j]$ $\Rightarrow$ contradiction of def of O_j ∎

Case 2: $\boxed{j \in O_j}$ $\quad p(j) = \begin{cases} \text{largest index } \underline{i<j} \text{ s.t. } i \,\&\, j \text{ do not conflict} \\ 0 \qquad\qquad\qquad \text{o/w} \end{cases}$

$n=6$ fix any $j \in [n]$

$p(1) = 0$
$p(2) = 0$
$p(3) = 1$
$p(4) = 1$
$p(5) = 2$
$p(6) = 2$

NOTE: (follow from def)

(i) $p(j)+1, \ldots, j-1$ conflicts with j

(ii) ~~$p(j), p(j)-1, \ldots, 1$~~ $1, 2, \ldots, p(j)$ does not conflict with j

i) $\Rightarrow$ if $j \in O_j \Rightarrow p(j)+1, \ldots, j-1$ cannot be picked.

(ii) $\Rightarrow O_j \setminus \{j\}$ is a valid schedule for $[p(j)]$.

Claim2: $O_j \setminus \{j\}$ is an optimal solution for $[p(j)]$

$$\Longrightarrow OPT(j) = v(O_j) = v_j + v(O_j \setminus \{j\}) \overset{\text{Claim2}}{=} v_j + OPT(p(j))$$

$$\Rightarrow \boxed{OPT(j) = v_j + OPT(p(j))}$$

$$\xrightarrow{\text{Overall}} \boxed{OPT(j) = \max\{\underbrace{OPT(j-1)}_{j \notin O_j}, \underbrace{v_j + OPT(p(j))}_{j \in O_j}\}}$$

Pf (idea) of Claim 2: By contradiction. $\exists\, O' \subseteq [p(j)]$ that is a valid schedule for $[p(j)]$ AND $v(O') > v(O_j \setminus \{j\})$

Note: $O' \cup \{j\}$ is a valid schedule for $[j]$

(follows from (ii))

$$v(O' \cup \{j\}) = v_j + v(O')$$
$$> v_j + v(O_j \setminus \{j\})$$
$$= v(O_j)$$

$\Rightarrow$ contradicts ~~the~~ def that O_j is optimal for $[j]$ $\blacksquare$