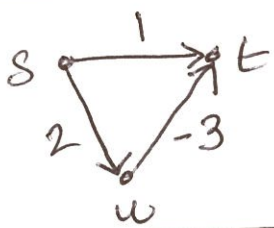
Nov 20

Shortest Path Problem

Input (•) Directed graph $G = (V, E)$
$\forall e \in E$, c_e ($c_e < 0$ is allowed)
BUT no negative cycle
(•) $t \in V$

Output : $\forall s \in V$, output a shortest s-t path

Attempt 1: Run Dijksta



shortest s-t path : s, u, t
but Dijksta picks : st

Attempt 2: Add some $\Delta > 0$ to all edge costs so that new costs $c'_e = c_e + \Delta \geq 0$
→ Run Dijksta

Doesn't work ! see T/F #7 on piazza (Above example with $\Delta = 3$ works)

UPSHOT: There is no known greedy or Divide & conquer algo for this problem

---

Assume (for now) : We only care about the cost of a shortest s-t path

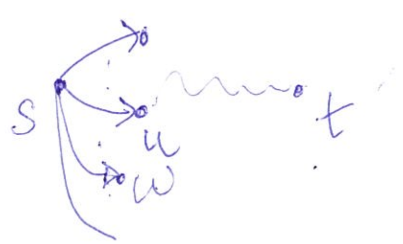
Goal: Design a ~~log~~ dynamic program

Attempt 3: $OPT(s)$ = cost of shortest s-t path $\forall s \in V$

(Q1) How many subproblems ? $OPT(s)$ $\forall s \in S$ ⇒ n sub-problem

(Q2) Recursive formula ($s \neq t$)

if shortest s-t path uses (s,u) as the first edge:

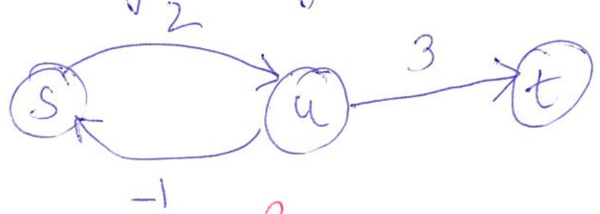


$$OPT(s) = c_{(s,u)} + OPT(u)$$

More generally:

recurrence
$$\boxed{OPT(s) = \min_{w:(s,w)\in E} \left\{ c_{(s,w)} + OPT(w) \right\}}$$

(Q3) Total ordering among sub-problems:



$$OPT(s) = \overset{2}{\cancel{c_{(s,u)}}} + OPT(u)$$

$$OPT(u) = \min\left\{ \underset{3}{\cancel{c_{(u,t)}}} + OPT(t),\ \underset{-1}{\cancel{c_{(u,s)}}} + OPT(s) \right\}$$

Cycle in dependence (specifically b/w u & s) $\Rightarrow$ X total ordering.

Solution: Introduce an (implicit) parameter to the sub-problem
not part of the problem def

$\Rightarrow$ have the parameter keep track of how "close" we are to t

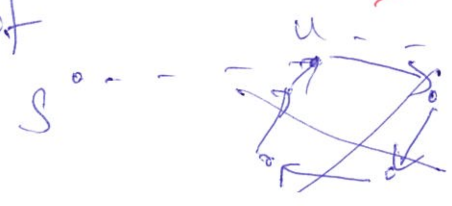
Attempt 4: See video from last year: gives a correct dynamic program but has exp. ~~rea~~ number of sub-problems.

Attempt 5: Bellman-Ford algo

$OPT(s, i)$ = cost of the shortest s-t path with $\leq i$ edges.
$0 \leq i \leq ?\ n-1$

Prop: If G has no negative cycle $\Rightarrow \forall s \in V$, $\exists$ a shortest s-t path that is simple.
$\Rightarrow$ always $\exists$ a shortest s-t path $\leq n-1$ edges

Pf (idea): If not



can drop this since total cost of cycle ≥ 0

Sub-problems: OPT(s, i) $\forall\, s \in V$

$\forall\, 0 \le i \le n-1$

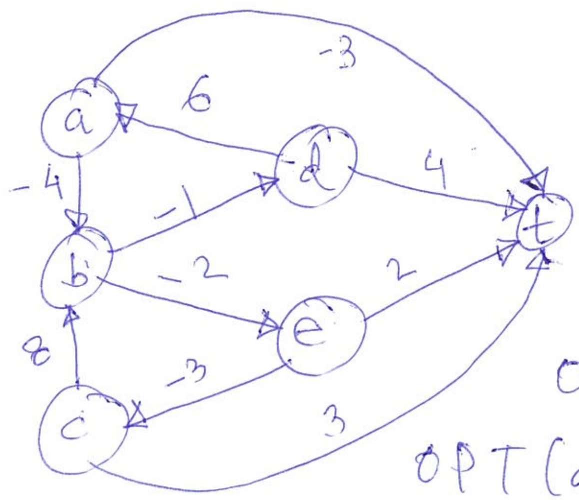
cost of shortest s-t path with $\le i$ edges in it

Goal: OPT(s, n-1)

$\forall s \in V$

(Q7) How many subproblems? $n \cdot n = n^2$

n=6



Let's focus on d (= s)

OPT(d, 0) = ∞ [d ≠ t]

OPT(d, 1) = 4 [d, t]

OPT(d, 2) = 6 − 3 [d, a, t]

OPT(d, 3) = 3 = 3 [d, a, t]

OPT(d, 4) = 6 − 4 − 2 + 2 = 2 [d, a, b, e, t]

n − 1 = 5 OPT(d, 5) = 6 − 4 − 2 − 3 + 3 = 0 [d, a, b, e, c, t]

OPT(d, 6) = OPT(d, 7) . . . = 0