Dec 8

RECALL:

(•) X is NP complete if

(i) $X \in NP$

(ii) $\forall Y \in NP$, $Y \leq_p X$

(•) Lemma 1: Let Y be NP-complete and $X \in NP$.

If $Y \leq_p X \Rightarrow X$ is NP-complete.

(•) General strategy to prove X is NP-complete:

(1) Show $X \in NP$

(2) Identify a known NP-complete problem Y

(typically $Y = 3\text{-SAT}$)

(3) Show $Y \leq_p X$

(•) THM 1: 3-SAT is NP-complete

THM 2: $3\text{-SAT} \leq_p IS$

COR 1: IS is NP-complete.

COR 2: VC is NP-complete (follows from $IS \leq_p VC$

Lemma 1 $\begin{cases} VC \in NP \\ + \text{Cor } 1 \end{cases}$

k-colorability (k-coloring)

$G = (V, E)$

<u>Def</u>: <u>k</u>-coloring $c : \underline{V} \to \{1, \ldots, k\}$ colors

if $\boxed{\forall (u,w) \in E \quad c(u) \neq c(w)}$

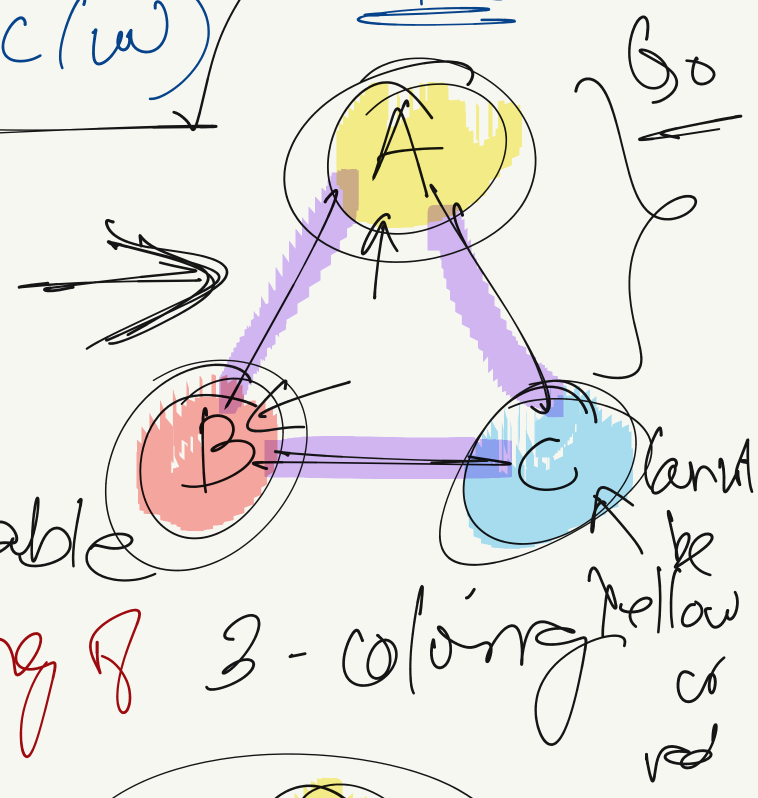
<u>Def</u> (k-colorability problem)

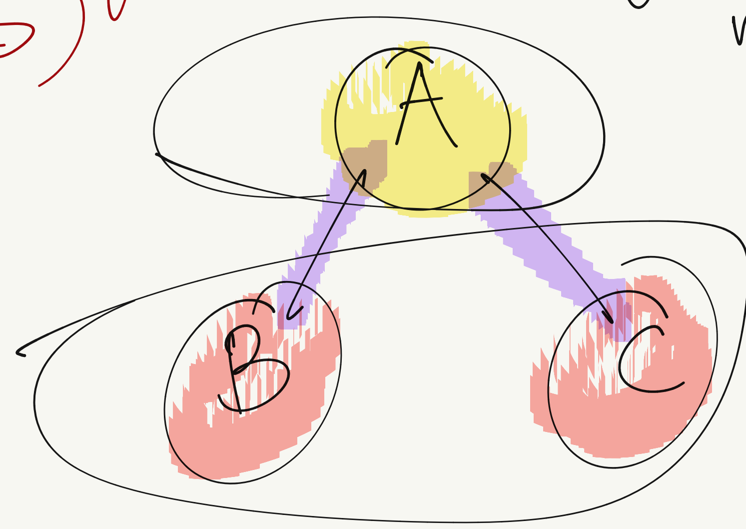
<u>Input</u>: $G = (V, E); k$

O/p: $\begin{cases} 1 & \text{if } G \text{ is } k\text{-colorable} \\ & (\exists \text{ a } k\text{-coloring of } G) \\ 0 & \text{o/w} \end{cases}$

$\to$ $G_0; 3$ ✓

$G_0; \underset{\underset{k}{\parallel}}{2}$ ✗

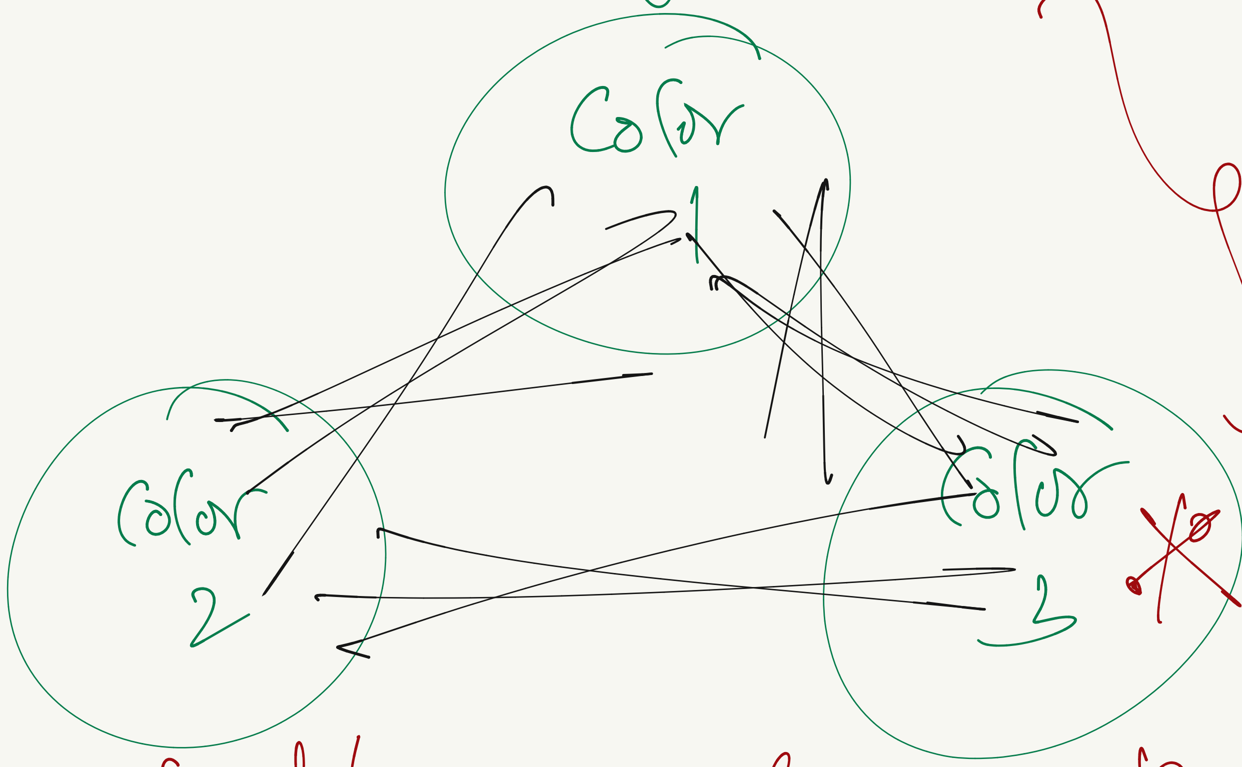




<u>Claim 1</u>: k-colorability $\in NP$

<u>pf (idea)</u>: witness a k-coloring.

3-colorable graph

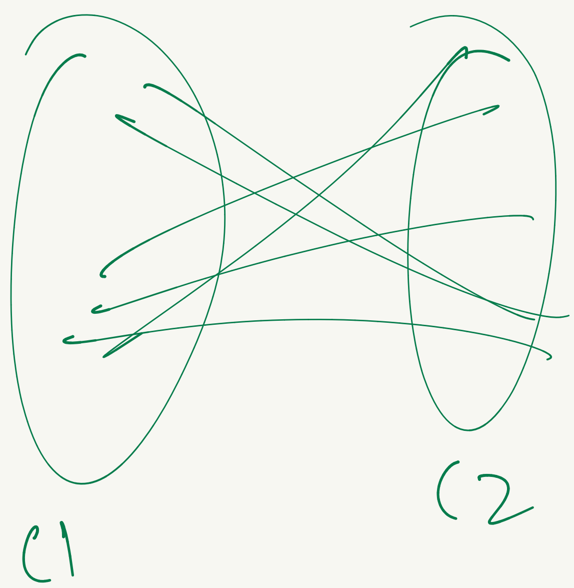


3-partite graph

k-colorable graph $\equiv$ k-partite.

2-colorable graph $\equiv$ 2-partite

bipartite



2-colorability $\in P$

(modify BFS to check if G is bipartite)

THM1: 3-coloring is NP-complete

THM2: $\boxed{\text{3-SAT} \le_p \text{3-coloring}} \le_p$ k-coloring ($k \geq 3$)

in book

+ 3-SAT is NP complete & 3-coloring $\in$ NP

Goal: 3-SAT $\le_p$ 3-coloring

Pf (idea): Given a 3-SAT formula

$C_i = t_1 \vee t_2 \vee t_3$

$\Phi = C_1, \dots, C_m$ on $x_1, \dots, x_n$ $\quad$ x_i or $\bar{x}_i$

$|\Phi|_{in} \Rightarrow \Theta(n+m) = N$

poly time (poly(n,m)) compute a graph G_Φ
s.t.

(i) $|G_\Phi| \le \text{poly}(n,m)$

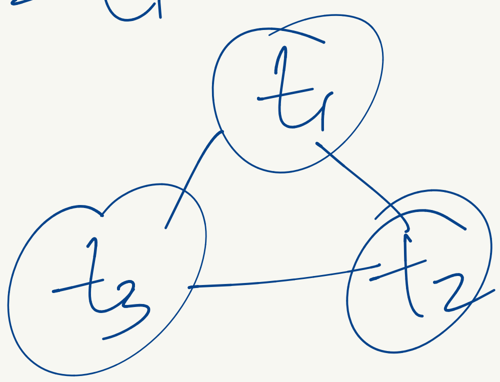
(ii) Φ is satisfiable $\iff$ G_Φ is 3-colorable

Redux: Algo Sat(Φ)

1. Convert Φ to G_Φ
2. $b \leftarrow$ Algo-3-coloring(G_Φ)
3. return b.

Idea: Use gadgets again ("Δ" will be more complicated)

In IS $C_i = t_1 \vee t_2 \vee t_3$

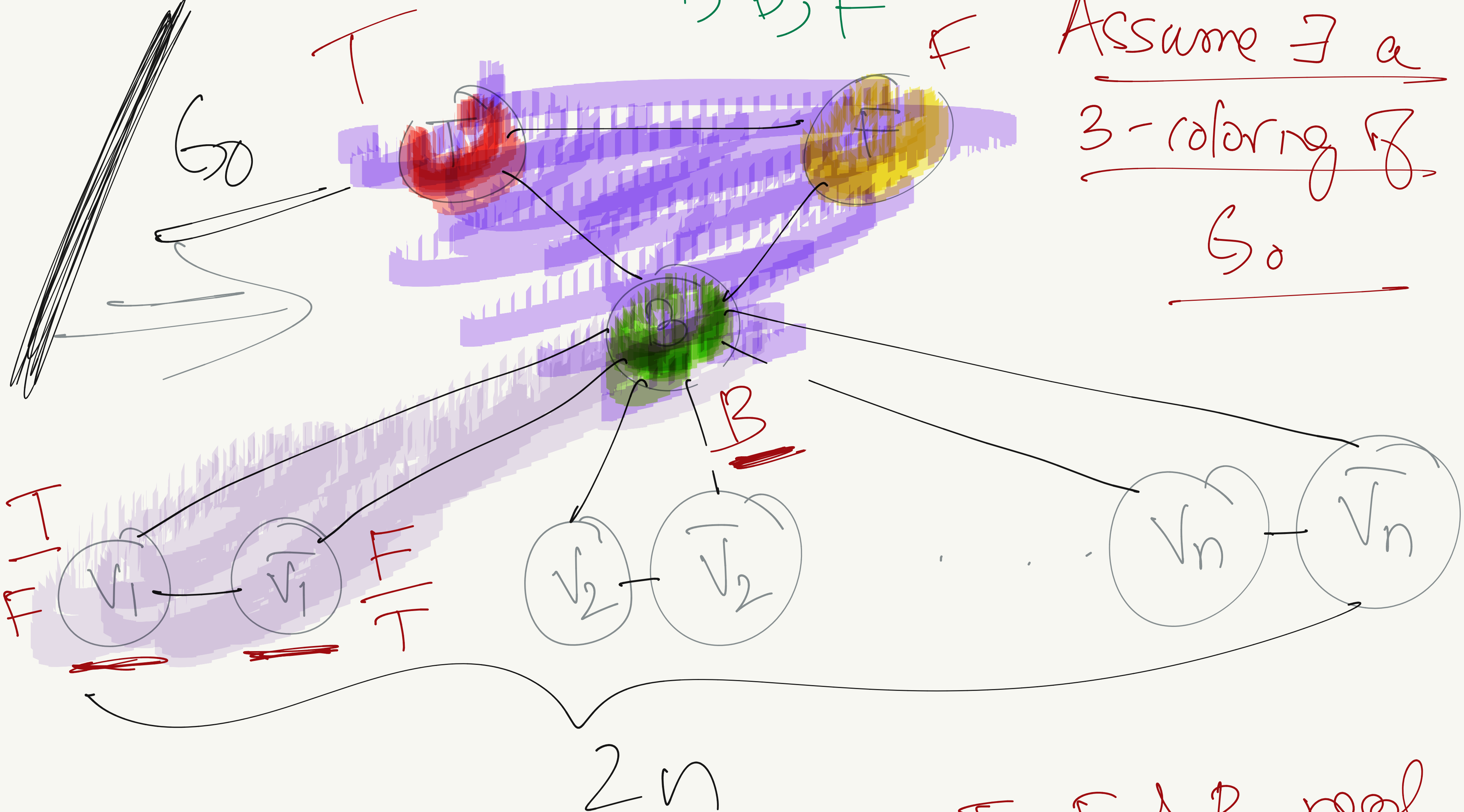


Step 1: Build a graph G_0 on $2n+3$ vertices

$2n \begin{cases} v_1, \dots, v_n & v_i \equiv x_i \\ \bar{v}_1, \dots, \bar{v}_n & \bar{v}_i \equiv \overline{x_i} \end{cases}$

3 special vertices: T, B, F

Assume $\exists$ a 3-coloring of G_0



In a 3-coloring of G_0 T, F & B need different colors

$c(T) = T$

$c(B) = B$

$c(F) = F$

3 coloring of $G \Rightarrow$ a truth assignment to $x_1, \dots, x_n$

Claim: $\exists$ a valid 3-coloring of G_0

$\iff$ for all i

$c(v_i) = T \wedge c(\bar{v}_i) = F$ $x_i \leftarrow T$

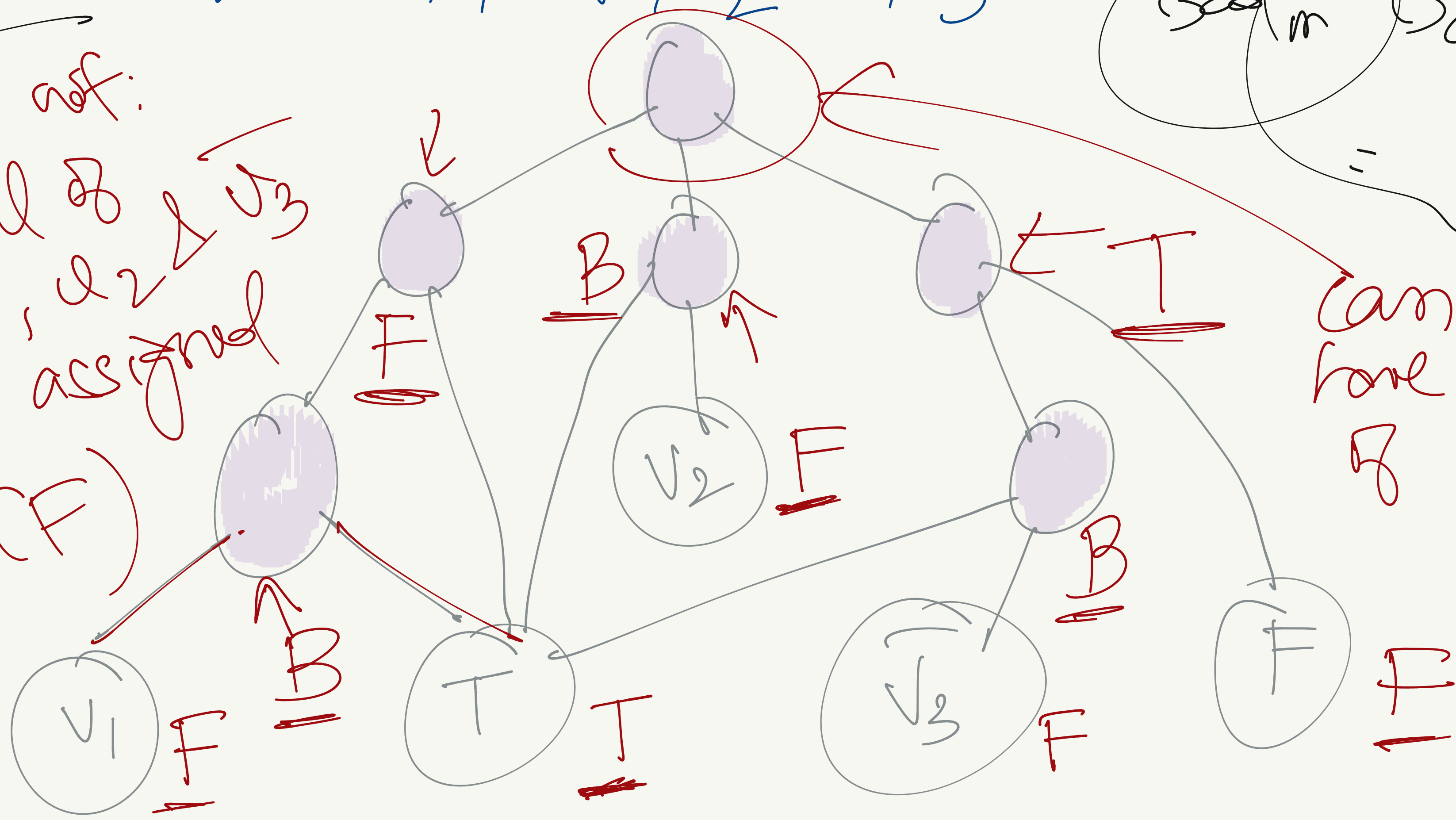
OR

$c(v_i) = F \wedge c(\bar{v}_i) = T$ $x_i \leftarrow F$

Step 2: Encode each clause C_i with its own gadget Gad_i to G_0

Ex: $C_i = x_1 \vee x_2 \vee \bar{x}_3$

Say all of v_1, v_2 & $\bar{v}_3$ are assigned $c(F)$



cannot have any of T, B or F

⇓

no 3-coloring

Claim: In a valid 3-coloring of Gad_i $(+G_0)$, at least one of $v_1, v_2, \bar{v}_3$ is assigned $c(T)$ (full proof in book)

$C_i = t_1 \vee t_2 \vee t_3$

t_1 t_2 t_3

⇒ Once we add all of $Gad_1, \ldots, Gad_m$ to G_0 ⟹ G_Φ

A sat. assignment to G_Φ ⟺ 3-coloring of G_Φ

Formal redux:

$\Phi = C_1, \ldots, C_m$ on $x_1, \ldots, x_n$

1. Construct G_0 on $\left\{ \begin{array}{l} v_1, \ldots, v_n \\ \bar{v}_1, \ldots, \bar{v}_n \end{array} \right\} \cup \{T, B, F\}$

2. Add G_i to G_0 for $i \in [m]$

Resulting graph G_Φ

THM: Φ is sat $\iff$ G_Φ is 3-colorable.

final exam material