Sep 13 **THEOREM:** For every input, (n, M, W, 2n pref lists) the Gale-Shapley (GS) outputs a stable mat

COROLLARY: Every input to the stable matching problem has a stable matching

(Pf): Follows from Theorem.

Pf of Theorem Fix an arbitrary input.

→ Say S is the output of the GS algo.

Want to prove: S is a stable matching.

Pf idea:

Lemma 0: for every input, GS algo terminates ($\leq n^2$ iterations)

Lemma 1: S is a perfect matching

Lemma 2: S has NO instability

Recall: stable matching = perfect matching + NO instability

Lem 0+1+2 ⇒ Theorem.

Pf (idea) Lemma 0: (see care package for proof details)

By algo design, in each iteration of GS algo, there is a new proposal.

$\Rightarrow$ #iterations = #proposals $\leq$ #pairs (w, m), $w \in W$, $m \in M$ $= |W \times M| = |W| \cdot |M| = n \cdot n = n^2$

$\Rightarrow$ #iterations $\leq n^2$ ∎

Obs 0: S is a matching (Ex: pf by inductions)

Obs 1: Once m gets engaged, he remains engaged to better woman.

Obs 2: If w proposes to m after m' $\Rightarrow$ m' > m in L_w

Lemma 3: If at the end of any iteration, w is free, then w has NOT proposed to ALL men.

Pf (idea) of Lemma 1: Goal: S is a perfect matching

Pf by contradiction (Use Obs 0, Lemma 0 + 3, Algo def.)

Pf details: Assume S is NOT a perfect matching

$\xRightarrow{\text{by Obs 0, algo definition}}$ $\exists$ a free woman w

w has not proposed to all men

$\xRightarrow{\text{by Lem 3}}$ $\exists$ a free woman who has not proposed to all men.

$\Longrightarrow$ Algo cannot have terminated

By condition in while loop $\longrightarrow$ contradicts Lemma 0. ∎