Oct 10

Lem 1: At the end of every iteration, if $u \in R$, then P_u is a shortest s-u path

P_u: (unique) s-u path in the "Dijkstra tree"

Pf (idea) By induction on $|R|$

Base case $|R| = 1 \Rightarrow R = \{s\}$ $\quad d(s) = 0$, $P_s = s$ ✓

I.H. Assume lemma is true for $|R| = k$ for some $k \geq 1$

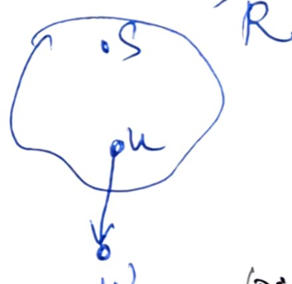
I.S. **Goal:** Prove the lemma for $|R| = k+1$

Let w be the $(k+1)^{th}$ vertex added to R

Claim: $\forall u \neq w \in R$, P_u is a shortest s-u path. (true by I.H.)

Goal: P_w is a shortest s-w path.

Assume w was added to Dijkstra tree because of the edge (u,w)



$$P_w = P_u, (u,w)$$

Claim: P_w is a shortest s-w path. $(*)$

Pf (idea) By contradiction

Assume $\exists$ s-w path P'_w s.t. $\ell(P'_w) < \ell(P_w)$

(Think of the state of the algo just <u>before</u> w is added.)

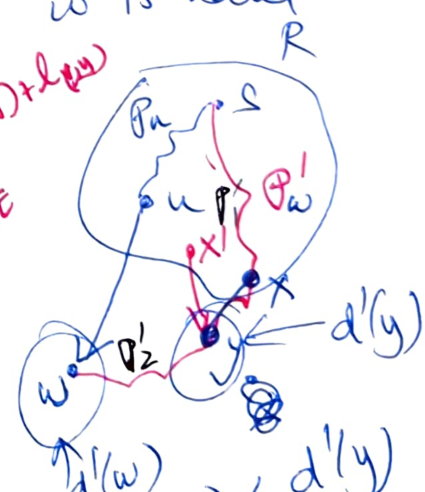
Since $w \notin R$ but $s \in R \Rightarrow P'_w$ has to "cross" R

$$P'_w = P'_1, (x,y), P'_2$$

$$d'(y) = \min_{\substack{v \in R \\ (v,y) \in E}} d(v) + \ell_{(v,y)}$$

$$\ell(P'_w) = \underbrace{\ell(P'_1)}_{\geq\, d(x)} + \ell_{(x,y)} + \ell(P'_2)$$

$$\geq \underbrace{d(x) + \ell_{(x,y)}}_{\geq\, d'(y)} + \ell(P'_2)$$



$$d'(w) \leq d'(y)$$

$$\geq d'(y) + \underbrace{l(P_2')}_{\geq 0}$$

$$\geq d'(y)$$

$$\geq d'(w)$$

$$= d(w) = l(P_w)$$

Algo def

$$\Rightarrow l(P_w') \geq l(P_w)$$

Contradicts (*) ∎