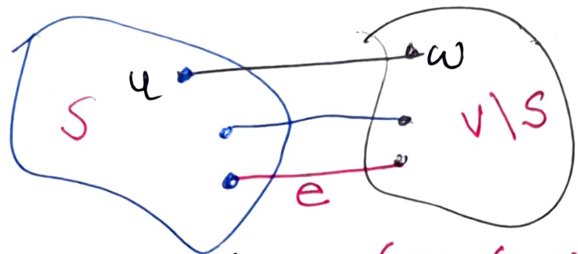
Oct 23

Assume: All c_e's are distinct (we'll later remove this assumption).

CUT PROPERTY LEMMA

For all cuts $(S, V\setminus S)$ s.t. $S \neq \phi$ and $V\setminus S \neq \phi$ $\equiv S \neq V$



Consider all crossing edges (i.e. $(u,w) \in E$ s.t. $u \in S$, $w \notin S$)

Let e be the cheapest crossing edge

$\Rightarrow$ e is present in ALL MSTs(!)

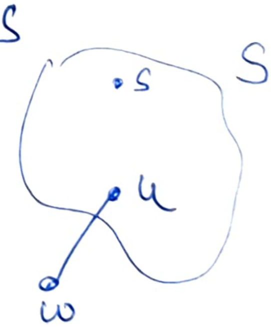
To prove correctness of Prim's / Kruskal's

Pf (idea) Any edge that is picked is the cheapest crossing edge for some cut.

Assume Cut Property Lemma (CPL) {+ all c_e's are distinct}

THM 1: Prim's algo is correct (i.e. $\forall$ inputs G, Prim's outputs an MST)

Pf (idea) Consider the state of Prim's when it is about to add the edge $e = (u,w)$ $u \in S, w \notin S$



Goal: e is the cheapest crossing edge for some cut $(S, V\setminus S)$

$\Rightarrow$ Pick S as in Prim's algo

Claim 1: $S \neq \phi$ $(u \in S)$

Claim 2: $S \neq V$ $(w \notin S)$

Claim 3: e is indeed the cheapest crossing edge for $(S, V\setminus S)$ (by definition of Prim's)

by CPL $\Rightarrow$ e is in all MSTs

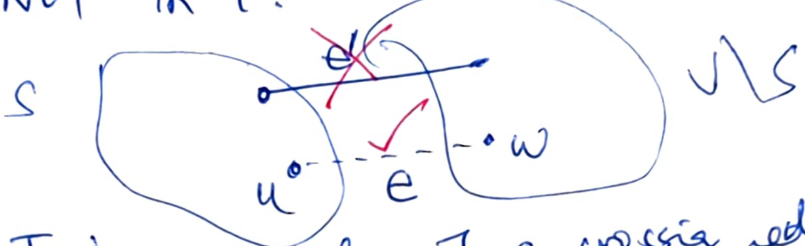
Claim 4: At the end of each iteration (S, T) is connected.
(Ex) $\Rightarrow$ after Prim's terminates, (V, T) is connected.

(Ex) Claim 1–4 + CPL $\Rightarrow$ Thm 1.

(*)

Pf (idea) of CPL: By contradiction.

Assume $\exists$ a cut $(S, V \setminus S)$ and an MST $T = (V, E')$
s.t. the ~~edg~~ cheapest crossing edge $e = (u, w)$, $u \in S$, $w \notin S$
is NOT in T.



Since T is connected, $\exists$ a crossing edge from T.
Let $e' \in E'$ be such a crossing edge.

Goal: show another spanning tree T' s.t. $c(T') < c(T)$

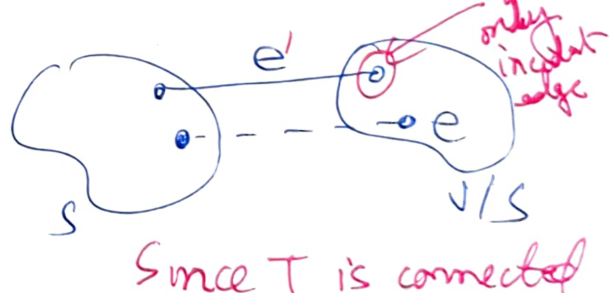
$$T' = (V, (E' \setminus \{e'\}) \cup \{e\})$$

Claim: $c(T') < c(T)$ Obs: $c_{e'} > c_e$

Pf: $c(T') = c(T) - c_{e'} + c_e$
Obs $\Rightarrow < c(T)$

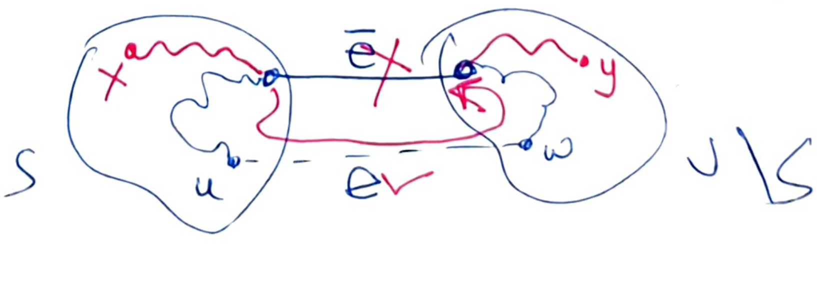
$\Rightarrow$ contradicts the assumption that T is an MST. e' is only incident edge

Q: where does the "proof" fail?
T' need not be connected!



Fix: Pick e' more carefully.

Since T is connected $\exists$ a u–w path P in T



since P starts at $u \in S$ & ends at $w \notin S$

$\Rightarrow \exists$ a crossing edge $\bar{e}$ on P.

$$T' = (V, (E' \setminus \{\bar{e}\}) \cup \{e\})$$

__Claim 1:__ $c(T') < c(T)$

__Claim 2:__ T' is connected.

Pf (idea) $x, y \in V$

__Case 1:__ $\exists$ an x-y path without using $\bar{e}$

__Case 2:__ x-y path uses $\bar{e}$ $\Rightarrow$ x, y still connected in T' {use the "scenic path"}

$\Rightarrow$ x, y are still connected $\square$

__Claim 1 + Claim 2__ $\Rightarrow$ T' is a spanning tree of cost $< c(T)$ $\Rightarrow$ contradicts our assumption that T is an MST $\square$