Nov 11

Simplified problem: Only want to compute the value of an optimal solution.

Assume: $f_1 \le f_2 \le \cdots \le f_n$

Def: $OPT(j)$ = value of an optimal solution $\underbrace{\text{for instance } [j]}_{(s_1, f_1, v_1), \ldots, (s_j, f_j, v_j)}$ $\forall j \in [n]$

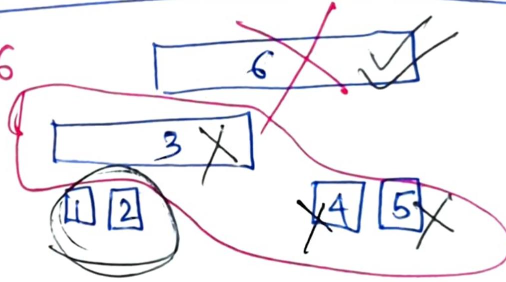
Goal: Compute $OPT(n)$

Assume: $OPT(0) = 0$

Def: $\forall j \in [n]$, let $\mathcal{O}_j$ be an optimal solution for $[j]$

$$\Rightarrow \quad OPT(j) \underset{(*)}{=} v(\mathcal{O}_j) = \sum_{i \in \mathcal{O}_j} v_i$$

Recursive algo 6×06



Fix $j \in [n]$

Case 1: $j \notin \mathcal{O}_j$

Claim 1: $\mathcal{O}_j$ is also optimal for $[j-1]$

$$\Rightarrow OPT(j) \underset{\text{by } (*)}{=} v(\mathcal{O}_j) \underset{\text{Claim 1}}{=} OPT(j-1) \Rightarrow \boxed{OPT(j) = OPT(j-1)}$$

Pf (idea) of Claim 1: Pf by contradiction

$\Rightarrow \exists$ a valid schedule $\mathcal{O}' \subseteq [j-1]$ s.t.

$$v(\mathcal{O}') > v(\mathcal{O}_j)$$

Since $[j-1] \subseteq [j]$
Since $\mathcal{O}'$ is valid schedule for $[j-1]$, it is also a valid schedule for $[j]$

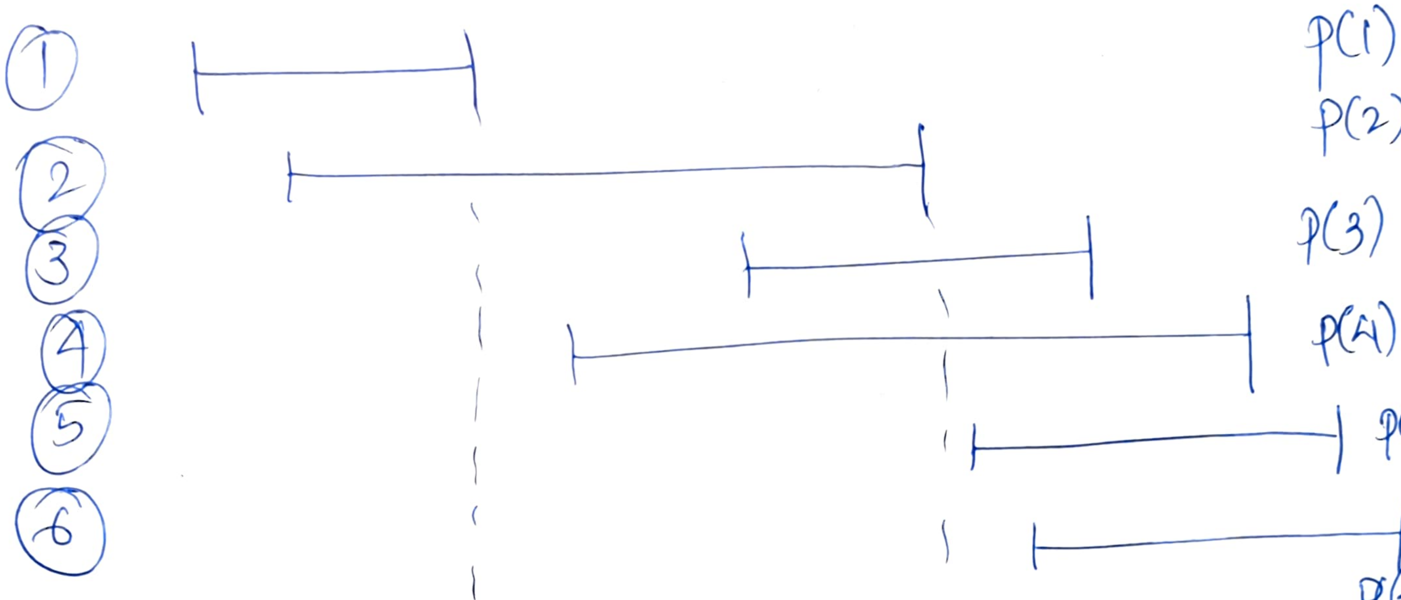
$\Rightarrow \mathcal{O}'$ is a valid schedule for $[j]$ AND $v(\mathcal{O}') > v(\mathcal{O}_j)$

$\Rightarrow \mathcal{O}_j$ is not an optimal solution for $[j]$
contradicts defn of $\mathcal{O}_j$ ■

Case 2: $j \in O_j$

Aside: $p(j) = \begin{cases} \text{largest index } i < j & \text{s.t. } i \,\&\, j \text{ do not conflict} \\ 0 & \text{o/w} \end{cases}$

$n = 6$



$p(1) = 0$

$p(2) = 0$

$p(3) = 1$

$p(4) = 1$

$p(5) = 2$

$p(6) = 2$

NOTE: (i) $p(j)+1, \ldots, j-1$ all conflict with j

follows from definition (ii) $1, \ldots, p(j)$ do <u>not</u> conflict with j

If $j \in O_j \Rightarrow$ remaining sub-problem $1 \ldots p(j)$

$\equiv [p(j)]$

$O_j \setminus \{j\}$ is a valid schedule for $[p(j)]$.

REMARKS:

(1) Can compute $p(1), \ldots, p(n)$ in $O(n \log n)$ time (Ex.)

(2) To compute $p(1), \ldots, p(n)$ you need $\Omega(n \log n)$ comparisons.

Claim 2: $O_j \setminus \{j\}$ is an optimal solution for $[p(j)]$

$$\Rightarrow OPT(j) = v_j + \cancel{OPT}\, v(O_j \setminus \{j\})$$

$$\overset{\text{claim 2}}{=} v_j + OPT(p(j))$$

$$\Rightarrow OPT(j) = v_j + OPT(p(j))$$

OVERALL:

$$\boxed{OPT(j) = \max \{ \overbrace{OPT(j-1)}^{j \notin O_j}, \overbrace{v_j + OPT(p(j))}^{j \in O_j} \}}$$

Pf of Claim 2: By contradiction. Assume $\exists$ a valid schedule $O' \subseteq [p(j)]$ s.t.

$$v(O') > v(O_j \setminus \{j\}) \quad (*)$$

Note: $O' \cup \{j\}$ is a valid schedule for $[j]$ ← $[j]$

by note (ii)

$$v(O' \cup \{j\}) = v(O') + v_j \overset{(*)}{>} v(O_j \setminus \{j\}) + v_j = v(O_j)$$

$\Rightarrow$ contradicts def of O_j ■