Subset Sum Problem

Nov 15

Input: n integer $w_1, \ldots, w_n$ s.t. $w_i > 0$ $\forall i$

Budget W

Output: A subset $S \subseteq [n]$ s.t.

(i) $\sum_{i \in S} w_i \le W$ (ii) max $w(S) = \sum_{i \in S} w_i$

Ex: $n = 3$ $w_1 = 1$, $w_2 = 3$, $w_3 = 3$

(i) $W = 7$, opt $= \{1,2,3\}$ as $w_1 + w_2 + w_3 = 1+3+3 = 7 \le 7 = W$

(ii) $W = 6$, opt $= \{2,3\}$

(iii) $W = 5$, opt $= \{1,2\}$ or $\{1,3\}$

sum of the number $= 4 < 5 = W$

Problem ver 2.0: compute max $w(S)$ over all S that satisfies (i)

Problem ver 3.0: max $|S|$

Greedy algo: sort the w_i's in non-decreasing order & pick as many as possible without going over W

Ex: Greedy algo pick the max $|S|$ $\Rightarrow$ solves ver 3.0 (Hint: Greedy stays ahead)

Example $W = 6$

greedy pick $\{1,2\}$ $\longrightarrow$ greedy is not optimal for ver 2.0

opt for ver 2.0 $\{2,3\}$

NOTE: There are no known greedy, D&C algo for problem 2.0

Dynamic Program:

Attempt: O_j be an optimal solution for subproblem $1..j$

$$OPT(j) = w(O_j)$$

Case 1: $j \notin O_j$ $\quad O_j$? $\quad 1..j-1$

Claim: O_j is still optimal $1..j-1$
(EX)

Case 2: $j \in O_j$

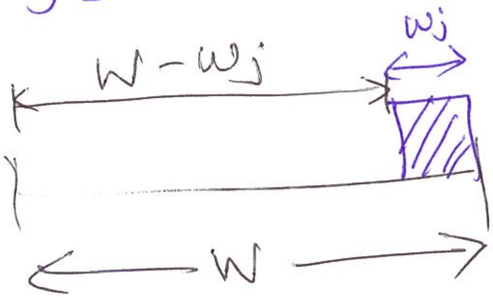
Q: What can we say about $O_j \setminus \{j\}$

Hope: Somehow argue $O_j \setminus \{j\}$ is optimal for $w_1..w_{j'}$ for some $j' < j$

Q: Why is there no hope (as detailed above?)

→ What is the subproblem left to solve once we pick w_j

$$w_1, .., w_j; W \longrightarrow w_1, .., w_{j-1}; W - w_j$$



Define subproblem: $w_1, .., w_j; B$ $\quad 0 \le j \le n$, $\quad 0 \le B \le W$

Def: $OPT(B, j)$ = wt of the opt. soln when the numbers are $w_1, .., w_j$ & budget is B

$0 \le j \le n$

$0 \le B \le W$

~~Case 2~~: $O(B, j)$ an opt soln

Case 0: $w_j > B \Rightarrow j \notin \mathcal{O}(B,j)$

$\Rightarrow OPT(B,j) = OPT(B,j-1)$

Case 1: $w_j \le B$ and $j \notin \mathcal{O}(B,j)$

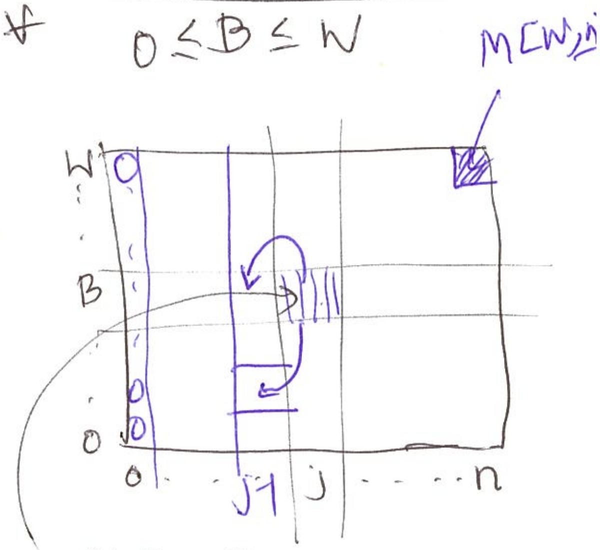
$OPT(B,j) = OPT(B,j-1)$

Case 2: $j \in \mathcal{O}(B,j)$ $(\Rightarrow w_j \le B)$

$OPT(B,j) = w_j + OPT(B-w_j, j-1)$

$$\Rightarrow \boxed{\begin{array}{l} \text{If } w_j > B \text{ then } OPT(B,j) = OPT(B,j-1) \\ \text{else} \\ OPT(B,j) = \max\{OPT(B,j-1), w_j + OPT(B-w_j, j-1)\} \end{array}} \;(*)$$

$OPT(B,0) = 0$ $\forall$ $0 \le B \le W$



$M[B,j] = OPT(B,j)$

Q1) Given $w_1, \dots, w_n; W$
if $M[B,j] = OPT(B,j)$ $\forall B, j$
what is the value we need to output?
$M[W,n]$

Q2) Init: $OPT(B,0) = 0$ $\forall B$

Q3) How many subproblems?
$(n+1)(W+1) = O(nW)$
$\le poly(n)$ If $W \le poly(n)$ ← assume this

Q4) Recurrence → $(*)$

Q5) Ordering among sub-problem
Compute the columns $M[0,:]$, $M[1,:]$ ...
$M[j-1,:]$, $M[j,:]$

SubsetSum $(w_1, \dots, w_n; W)$

0. Allocate $(W+1) \times (n+1)$ matrix M
1. $M[B, 0] \leftarrow 0 \quad \forall\ 0 \le B \le W$
2. for $j = 1 .. n$
    - for $B = 0 .. W$
        - if $w_j > B$ then $M[B, j] \leftarrow M[B, j-1]$
        - else $M[B, j] \leftarrow \max\{ M[B, j-1],\ w_j + M[B - w_j, j-1] \}$
3. Return $M[W, n]$

Prove correctness by induction (Ex.)