Nov 20

Ideas from Monday:

→ When defining subproblems, introduce an implicit parameter

→ Specifically, we'll use the implicit param to keep track of how "close" we are to t.

Attempt 5: Bellman-Ford Algo

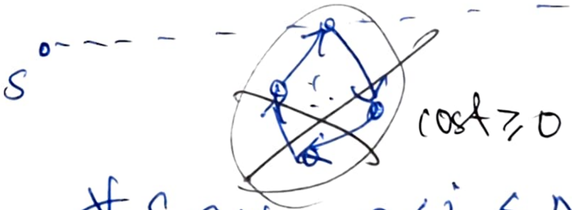
$OPT(s, i)$ = cost of shortest s-t path with $\leq i$ edges.

$\forall s \in V$

$\forall\ 0 \leq i \leq n-1$

Prop: ~~If~~ G has no negative cycle $\Rightarrow \forall s \in V, \exists$ a shortest s-t path that is simple $\Rightarrow \exists$ shortest s-t path $\leq n-1$ edges

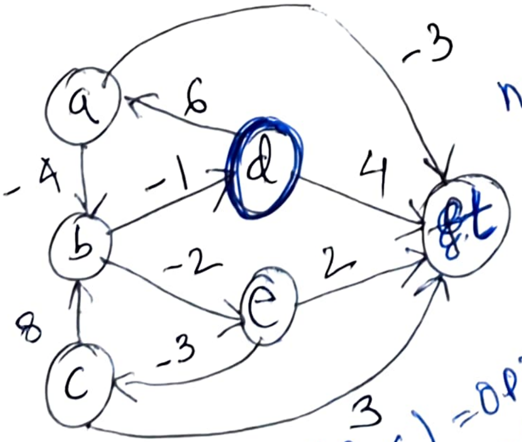
(Pf (idea): ~~If~~ not.



Sub-problems: $OPT(s, i)$ $\quad \forall s \in V, 0 \leq i \leq n-1$

Goal: $OPT(s, n-1)$ $\quad \forall s \in V$ {follows from defn of $OPT(s,i)$ + Prop}

Note: $OPT(s, i+1) \leq OPT(s, i)$



focus on $s = d$

$n-1 = 5$

$OPT(d, 0) = \infty$ (since $d \neq t$)

$OPT(d, 1) = 4$ $\quad [d, t]$

$OPT(d, 2) = 6 - 3 = 3$ $\quad [d, a, t]$

$OPT(d, 3) = 3$ $\quad [d, a, t]$

$OPT(d, 4) = 6 - 4 - 2 + 2 = 2$ $\quad [d, a, b, e, t]$

$OPT(d, 5) = 6 - 4 - 2 - 3 + 3 = 0$ $\quad [d, a, b, e, c, t]$

$OPT(d, 6) = OPT(d, 5) = 0$

(Q1) $OPT(s,i)$ = cost of shortest s-t path $\le i$ edges

(Q1) How many subproblems? $n \times n = n^2$

(Q2) Recursion

$OPT(t,0) = 0$ $\qquad$ $OPT(u,0) = \infty$ $\forall u \neq t$

Consider $OPT(u,i)$ $\quad i > 0$

Case 1: $\exists$ a shortest s-t path (with $\le i$ edges) that actually uses $\le i-1$ edges.

$\Rightarrow OPT(u,i) = OPT(u,i-1)$

Case 2: All shortest s-t paths use $= i$ edges

$\Rightarrow \exists$ a "first" (u,w) edge

$OPT(u,i) = c_{(u,w)} + OPT(w,i-1)$

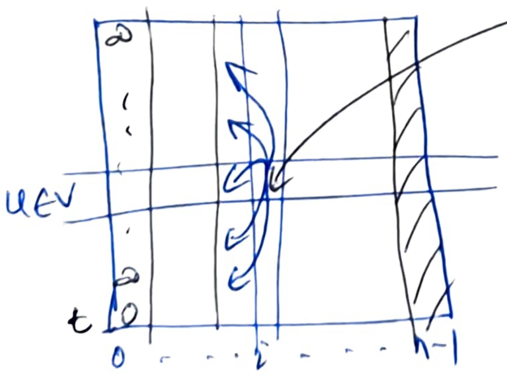
In general

$$OPT(u,i) = \min_{\substack{x: \\ (u,x)\in E}} \left\{ c_{(u,x)} + OPT(x,i-1) \right\}$$

OVERALL: $i > 0$

$$\boxed{OPT(u,i) = \min \Big\{ \underbrace{OPT(u,i-1)}_{\text{case 1}},\ \underbrace{\min_{\substack{x: \\ (u,x)\in E}} \{ c_{(u,x)} + OPT(x,i-1) \}}_{\text{case 2}} \Big\}}$$

$M[u,i] = OPT(u,i)$



Bellman-Ford Algo

$O(n^2)$

(0) Allocate an $n \times n$ matrix M

(1) $M[t,0] \leftarrow 0$, $M[u,0] \leftarrow \infty$ $\forall u \neq t$ $\leftarrow O(n)$

(2) $O(n^3)\left\{ \begin{array}{l} \text{for } i = 1 \cdot\cdot n-1 \\ \quad \text{for } u \in V \end{array} \right\} \leq n^2$

~~$M[u,i] \leftarrow \min$~~

$O(n)\{M[u,i] \leftarrow \min\{M[u,i-1], \min_{x:(u,x)\in E}\{c_{(u,x)} + M[x,i-1]\}\}$

(3) Return $M[s, n-1]$ $\forall s \in V$ overall: $O(n^3)$

$O(n)$ (See book: $O(mn)$)