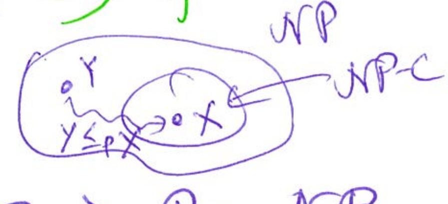
Dec 4

RECAP

Def: X is NP-Complete (or NP-C) if
(i) $X \in NP$
(ii) $\forall Y \in NP, \quad Y \leq_p X$



Lemma: Let X be NP-C. If $X \in P \Rightarrow P = NP$.

$\rightarrow X$ is "hard" $\equiv X$ is NP-complete

Satisfiability / SAT problem

General: SAT formula on n variables $X = \{X_1, .., X_n\}$ ← Boolean
↳ AND clauses
↳ OR of literals
↳ either X_i, $\overline{X_i}$

Ex: $\Phi_0^{(X_1, X_2, X_3)} = \underbrace{(X_1 \vee \overline{X_2})}_{\text{clause}} \wedge \underbrace{(\overline{X_1} \vee \overline{X_3})}_{\text{Clause}} \wedge (X_2 \vee \overline{X_3})$

literal

Generally: $C_1 \wedge \cdots \wedge C_m$ $\quad C_i$ clause

$\equiv C_1, \ldots, C_m$

Clause C_i = OR of literals $\quad t_1 \vee t_2 \vee \cdots \vee t_\ell$

each $t_i \in \left\{ \begin{matrix} X_1, X_2, \ldots, X_n \\ \overline{X_1}, \overline{X_2}, \ldots, \overline{X_n} \end{matrix} \right.$

Assignment: $v: X \rightarrow \{0,1\}$

$X_1 =$	$\boxed{0}$	1	0
$X_2 =$	$\boxed{0}$	1	0
$X_3 =$	$\boxed{0}$	1	1

① $\Phi_0 \overset{X_1\ X_2\ X_3}{(0, 0, 0)} = (0 \vee \overline{0}) \wedge (\overline{0} \vee \overline{0}) \wedge (0 \vee \overline{0})$

$= (0 \vee 1) \wedge (1 \vee 1) \wedge (0 \vee 1)$

$= 1 \wedge 1 \wedge 1 = 1$

$(0,0,0)$ satisfies Φ_0

↳ 3-SAT verifier (Φ, v)

→ Evaluates Φ on v & outputs the eval.

Property verifier:

If $\Phi \in$ 3-SAT, $\exists$ a ~~satisfying~~ assignment v s.t. 3-SATverif $(\Phi, v) = 1$

If $\Phi \notin$ 3-SAT, $\forall$ assignment $\bar{v}$ 3-SATverifier $(\Phi, \bar{v}) = 0$

Lemma 2: If Y is NP-C AND $Y \leq_p X$ AND $X \in NP \Rightarrow X$ is NP-C

Pf: Need to show

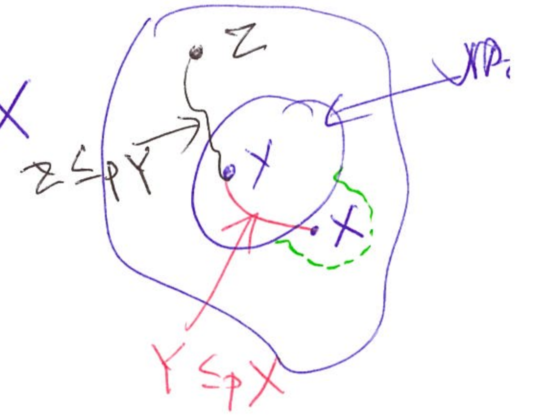
(i) $X \in NP$

✓ (ii) $\forall Z \in NP$, $Z \leq_p X$

As Y is NP-C, $\boxed{\forall Z \in NP,}$

$Z \leq_p Y \leq_p X$

$\Rightarrow Z \leq_p X$ ∎



General strategy to prove a new problem X is NP-C

(i) $X \in NP$

(ii) $\exists$ an NP-C problem Y s.t. $Y \leq_p X$

$\Rightarrow$ X is NP-C

Lem 2

In "practice" Y is typically 3-SAT.

Goal: IS is NP-C

Seen: IS $\in$ NP

② $\Phi_0(1,1,1) = (1 \vee \overline{T}) \wedge (\overline{T} \vee \overline{T}) \wedge (1 \vee \overline{T})$

$= (1 \vee 0) \wedge (0 \vee 0) \wedge (1 \vee 0)$

$= 1 \wedge 0 \wedge 1$

$= 0$

(1,1,1) does not satisfy Φ_0 / SAT formula

Def: An assignment satisfies Φ if Φ evaluates to TRUE/1 given the assignment

$\equiv$ ALL ~~cla~~ clauses evaluate to TRUE/1 on the assignmt.

SAT problem / Satisfiability problem

Input: SAT formula Φ

Output: $1 \iff \exists$ a satisfying assignment to Φ

Ex. i/p (1): $\Phi_1 = (\underset{1}{X_1} \vee \overline{X_2} \vee X_4), (\overline{X_1} \vee \underset{0}{\overline{X_3}} \vee \underset{1}{X_5})$

o/p (1): 1 ← 3-SAT

~~Xo~~ $(X_1 = 1, X_3 = 0, ?)$

i/p (2): $\Phi_2 = X_1, \overline{X_1}$ $(\equiv X_1 \wedge \overline{X_1} = 0)$

o/p (2): 0 ✗ 3-SAT

3-SAT formula is a SAT formula $C_1, \ldots, C_m$ s.t each C_i has exactly 3 literals

3-SAT problem

input: 3-SAT formula Φ output: $1 \iff \exists$ a satisfying assignment to Φ

Lem 1: 3-SAT $\in \mathcal{NP}$ Pf (idea): witness assignment.

THM 1: 3-SAT is $\mathcal{NP}$-C (see book for proof)

THM 2: $3\text{-SAT} \le_p IS$

Pf (idea): Given a 3-SAT formula $\Phi = C_1, \dots, C_m$

$\xrightarrow{\text{redux}} G_\Phi \, ; \, m$

s.t.

Φ is SAT $\iff (G_\Phi, m) \in IS$