Dec6

RECAP

(•) X is NP-C if
(i) $X \in NP$ (ii) $\forall Y \in NP,\ Y \leq_p X$

(•) Lemma 1: Let Y ~~X~~ be NP-C and $X \in NP$.
If $Y \leq_p X \Rightarrow X$ is NP-C

(•) General strategy to prove X is NP-C
(1) Show $X \in NP$
(2) Identify a known NP-C problem Y (typically 3-SAT)
(3) Show $Y \leq_p X$

(•) THM 1: 3-SAT is NP-C
THM 2: 3-SAT $\leq_p$ IS (today!)
COR 1: IS is NP-C

COR 2: VC is NP-C (since IS $\leq_p$ VC and VC $\in NP$)

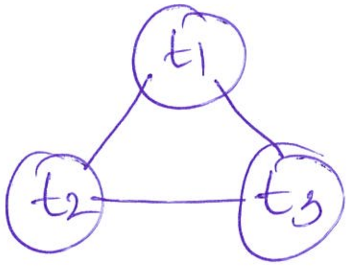
Pf (idea) of THM 2: Input: 3-SAT Φ $C_1, \ldots, C_m$
$\hookrightarrow$ compute G_Φ
s.t Φ is satisfiable $\Longleftrightarrow$ G_Φ has an IS of size m

$$\Phi \rightarrow \boxed{\text{Pre-proc}} \xrightarrow{(G_\Phi,\, m)} \boxed{\text{Algo for IS}} \xrightarrow{0/1} \text{same } 0/1$$

Redux idea: Use a "gadget"
2 equiv ways of saying 3-SAT formula Φ is satisfiable

(1) Assign 0/1 to each $X_1, \ldots, X_n$ s.t the assignment satisfies ≥ 1 literal in each C_i
(2) Pick a literal from each clause s.t you do NOT pick BOTH X_i & $\overline{X_i}$ for some i.

(Gadget) $C = t_1 \vee t_2 \vee t_3$



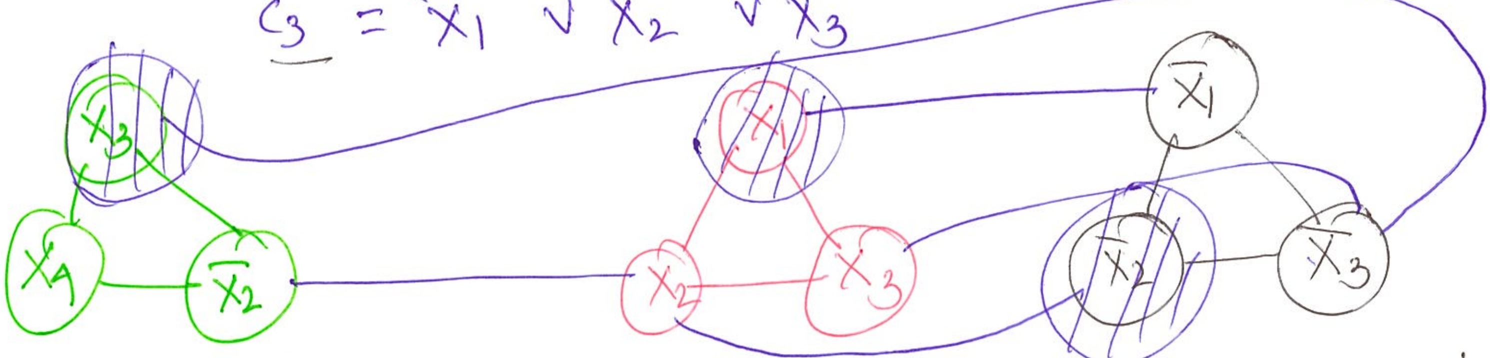
$IS = \{\}$, $\{t_1\}$ $\{t_2\}$ $\{t_3\}$

Each non-empty IS $\equiv$ picking a literal from C

$n=4$
$m=3$

$c_1 = x_1 \vee x_2 \vee x_3$ $\quad c_2 = \overline{x_2} \vee x_3 \vee x_4$

$c_3 = \overline{x_1} \vee \overline{x_2} \vee \overline{x_3}$



Step 1: For each clause, create its own "Δ"

Step 2: Add an edge between any pair x_i & $\overline{x_i}$ s.t they appear in different clauses

gives G_Φ

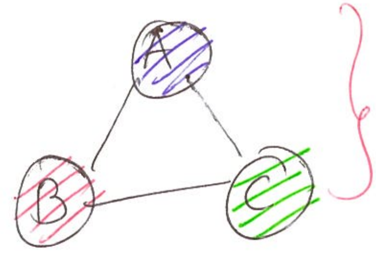
$\{x_1, \overline{x_2}, x_3\}$ is an IS $\equiv$ $x_1 = 1$, $x_2 = 0$, $x_3 = 1$

more generally: an IS in G_Φ of size m $\equiv$ satisfying assignment for Φ

THM: Φ is SAT $\iff$ G_Φ has an IS of size $\geq m$.

k-colorability (k-coloring) HW 8, Q3

$G = (V, E)$



G_0

3-colorng

Def: k-colorng of G

is $c: V \longrightarrow \underbrace{\{1, \ldots, k\}}_{\text{k-colors}}$

s.t.

$\forall (u, w) \in E, \quad c(u) \neq c(w)$

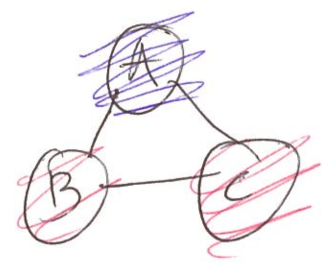
$G_0, 3 \quad 1$

$G_0, 2 \quad 0$

Def: k-colorability problem

Input: G, k

o/p: $\begin{cases} 1 & \text{if } G \underbrace{\text{has a k-coloring}}_{G \text{ k-colorable}} \\ 0 & \text{o/w} \end{cases}$

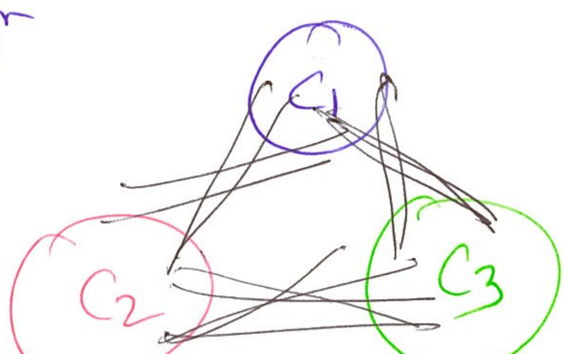


Claim: k-colorability $\in NP$

Pf (idea) Witness k-colorng

3-colorable gr

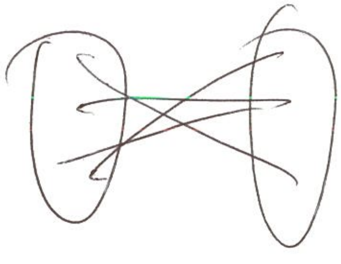
3-partite graph



k-colorable graph $\equiv$ k-partite graph

2-colorabe graph $\equiv$ 2-partite graph

bipartite graph.



2-colorability $\in P$

(modify BFS to check if G is bipartite)

THM3: 3-coloring is NP-C

THM4: $\boxed{\text{3-SAT} \le_p \text{3-coloring}} \le_p$ k-coloring ← see the book

$k \geq 3$

+ 3-SAT is NP-C + 3-coloring $\in$ NP

Goal: 3-SAT $\le_p$ 3-coloring

Pf (idea) Given a 3-SAT formula

$\Phi = C_1, \dots, C_m$ $\qquad C_i = t_1 \vee t_2 \vee t_3$

on $X_1, \dots, X_n$

$|\Phi| = \Theta(m+n)$

in poly $(n+m)$ time generate a graph G_Φ

s.t. Φ is satisfiable $\Longleftrightarrow$ G_Φ is 3-colorable

Redux: AlgoSat (Φ)

1. Construct G_Φ from Φ
2. $b \leftarrow$ Algo-3-color (G_Φ)
3. Return b