

Sep 11

THEOREM! For every input to SMP ( $n, M, W, 2n$  pref lists)  
Gale-Shapley (GS) algo outputs a stable matching.

COROLLARY! Every input to the SMP has  $\geq 1$  stable matching.

Pf. Follows from THEOREM

Pf of THEOREM! Fix an arbitrary input  
→ Let's say  $S$  is the output of the GS on  
Want to prove:  $S$  is a stable matching.

Pf idea: Lemma 0! For every input, the GS algo terminates.

Lemma 1:  $S$  is a perfect matching

Lemma 2:  $S$  has no instability.

Lemmas 0, 1, 2  $\Rightarrow$  THEOREM is true

Pf (idea) for Lem 0: (see core package for proof details)

By algo design, in each iteration of GS algo, there is a new propose

$$\begin{aligned} \Rightarrow \# \text{ iterations} &= \# \text{ proposals} \leq \# \text{ pairs } (w, m) = |W \times M| \\ &= |W| \cdot |M| \\ &= n \cdot n \\ &= n^2 \end{aligned}$$

Obs 0:  $S$  is a matching (algo def.)

Obs 1: Once  $m$  gets engaged, he keeps getting engaged to better women

Obs 2: If  $w$  proposes to  $m$  after  $m'$ ,  $m' > m$  in  $L_w$

Lemma 3: If at the end of any iteration,  $w$  is free  
 $\Rightarrow w$  has proposed to ALL men.

Pf (idea) of Lem 1: