Sep 21 **Def:** An undirected graph $T=(V,E)$ is a tree if (i) T is connected and (ii) T has no cycles

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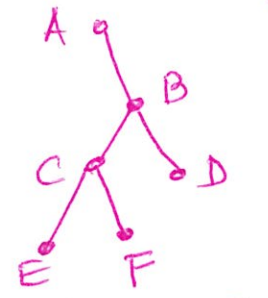
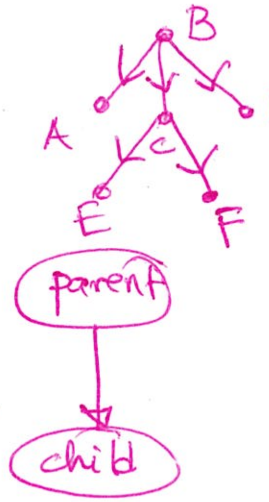


$n=6, m=5$

THM1: Any tree on n nodes has exactly $n-1$ edges.

Pf idea: Pick any vertex $r \in V$

1. Root T at r
2. Direct edges "away" from r
3. Argue every (directed edge) has a unique child
4. Argue that r is the only node that is not the child of ~~any~~ some (directed) edge.



$\Rightarrow |E| = |V \setminus \{r\}| = n-1$

for any sets A, B $\quad A \setminus B \stackrel{def}{=} \{a \in A \mid a \notin B\}$

Pf details: Pick an $r \in V$ and root T at r.

Ex: For every edge $(u,w) \in E$, one of u or w is closer (with ~~respect to~~ distance) to r than the other

↳ Direct edge from closer node to the farther node

Ex → **Claim 1:** r is not the child of any (directed) edge.

Ex → **Claim 2:** Every (directed) edge has a unique child

Next ↓

Claim 3: Every non-root vertex is child of ≤ 1 (directed edge)

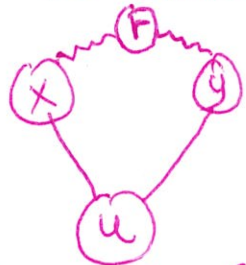
Ex → **Claim 4:** Every non-root vertex is the child of some (directed) edge.

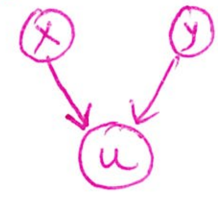
Claims 1, 2, 3, 4 $\Rightarrow \exists$ a 1-to-1 correspondence (bijection) between E and $V \setminus \{r\}$ $\Rightarrow |E| = |V \setminus \{r\}| = n-1$ ← except for Claims 1–

Pf (idea/details): Pf by contradiction.

For the sake of contradiction, assume $u \in V \setminus \{r\}$ is the child of 2 (directed) edges

Now consider T:





since T is connected

$\Rightarrow \exists$ an r-x path & an r-y path

note $x, u, y \sim r \sim x$ is a cycle $\Rightarrow$ contradicts that T has no cycles $\blacksquare$

THM 2: Let T be an undirected graph. Then ANY 2 of them imply the 3rd:

1. T is connected
2. T has no cycles
3. T has $n-1$ edges

THM 1: 1 + 2 $\Rightarrow$ 3

1 + 3 $\Rightarrow$ 2
2 + 3 $\Rightarrow$ 1 } Ex.