

Constructing dependent random probability measures from completely random measures

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Introduction

Two strands of research in NPBayes modelling of random probability measures (RPMs):

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- e.g. power-law behaviour or more uncertainty on number of clusters:
Normalized Random Measures [James et al., 2005, Kingman, 1975]
(e.g. normalized generalized Gamma process)
Poisson-Kingman processes [Pitman, 2003]
(e.g. Pitman-Yor process [Pitman and Yor, 1997])

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- for data violating the assumption of exchangeability:
Time-series, spatial data, conditional density modelling
- Research traces back to work of [MacEachern, 1999] on *dependent RPMs*

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This talk:

Flexible constructions for dependent RPMs with flexible marginals

Relevant work

There is a rich literature on dependent RPMs

- the seminal work of [MacEachern, 1999] on dependent DPs
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- [C. Chen, V. Rao, W. Buntine and Y.W. Teh \(2013\)](#)
Dependent Normalized Random Measures

Completely random measures (CRMs)

- A random measure μ on some space (\mathbb{X}, Σ_X) such that

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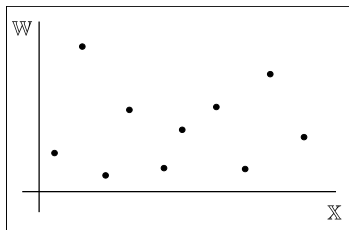
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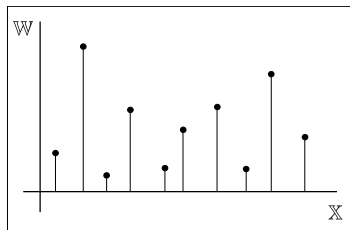
- (x_i, w_i) : events of a Poisson process on the space $\mathbb{X} \times \mathbb{W}$, where $\mathbb{W} = [0, \infty)$.
- The Poisson process has intensity $\nu(w, x) = \rho(w)h(x)$, where $\rho(w)$ is the Lévy intensity of the CRM, and $h(x)$ is the base probability density.

Normalized random measures



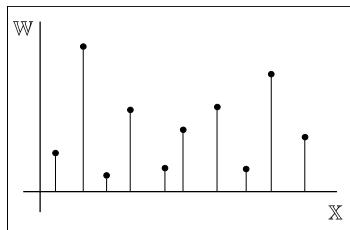
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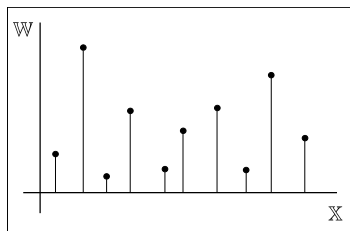
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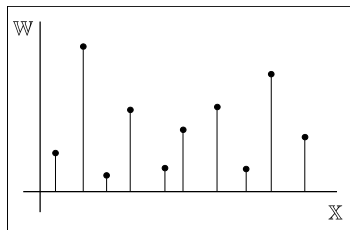
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We want: **Dependent normalized random measures, G_t**

Dependent normalized random measures

- Define a common latent CRM/Poisson process.
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- Define dependent measures via transformations of this process.
 - ▶ Superposition [Rao and Teh, 2009, Griffin et al., 2013]
 - ▶ Rescaling
 - ▶ Thinning [Lin et al., 2010, Lin and Fisher, 2012]

Normalize these dependent CRMs to produce dependent NRMs.

Superposition theorem

- The superposition of two independent Poisson processes with intensity $\nu_i(\cdot)$, $i = 1, 2$ is a Poisson process with intensity $\nu_1(\cdot) + \nu_2(\cdot)$
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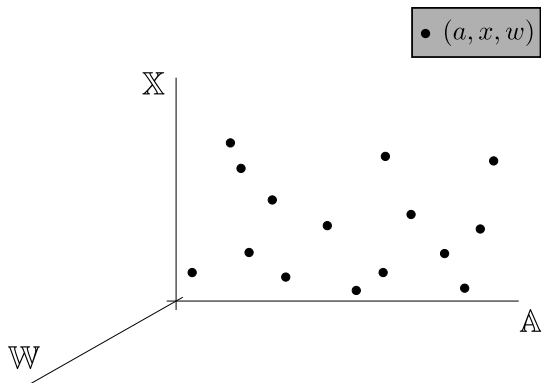
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- If $\nu(\cdot)$ factors as $\rho(w)h(x)\nu_a(a)$, then the resulting CRM has Lévy intensity $(\int_{\mathbb{A}} \nu_a(a)da) \rho(w)h(x)$.

Spatial Normalized Gamma processes [Rao and Teh, 2009]

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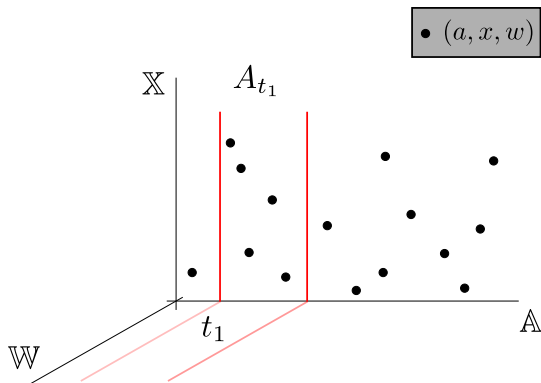
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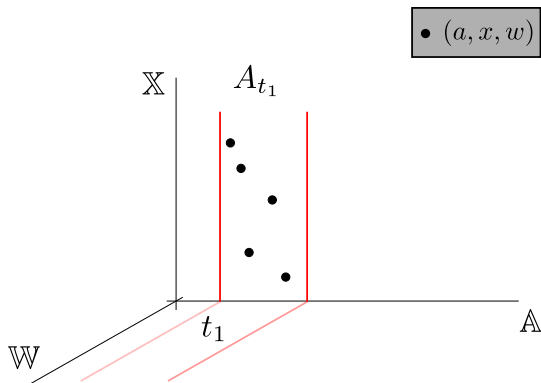
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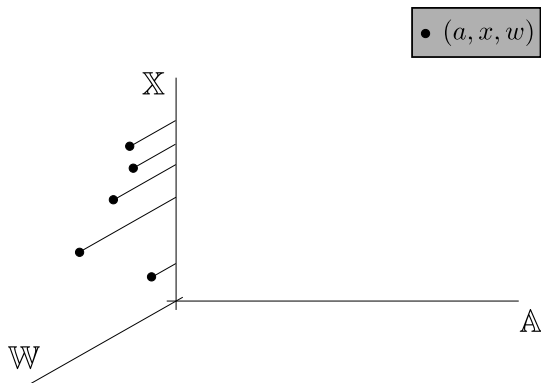
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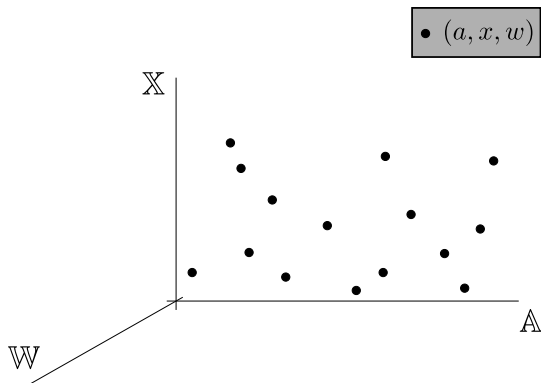
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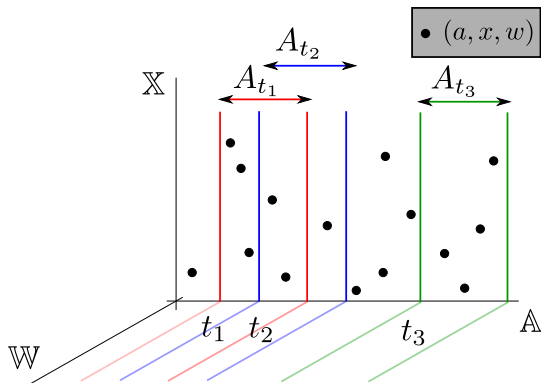
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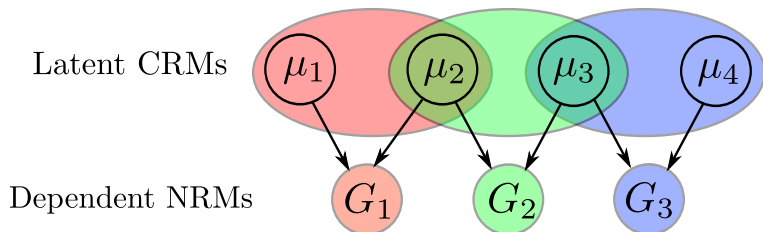
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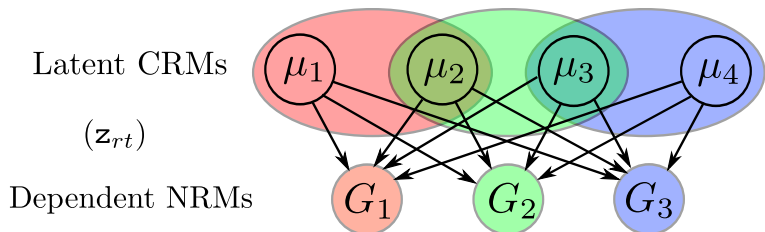


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- Dependency across NRMs is controlled by amount of overlap of A_t 's

Dependent normalized random measures ([Griffin et al., 2013])

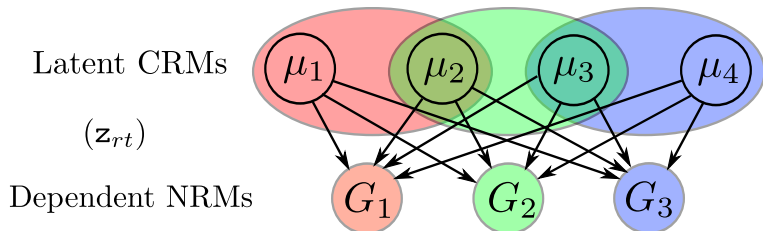


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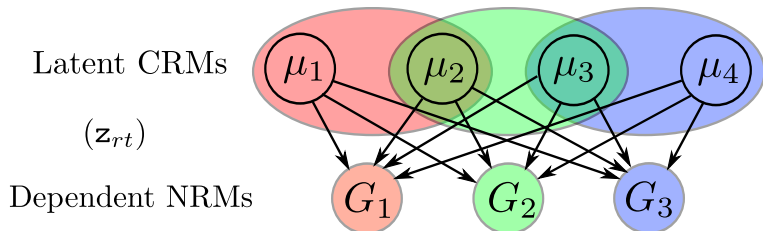
$$G_t \propto \sum_{r=1}^R z_{tr} \mu_r, \quad z_{tr} \in \{0, 1\} \quad (1)$$

Mixed normalized random measures



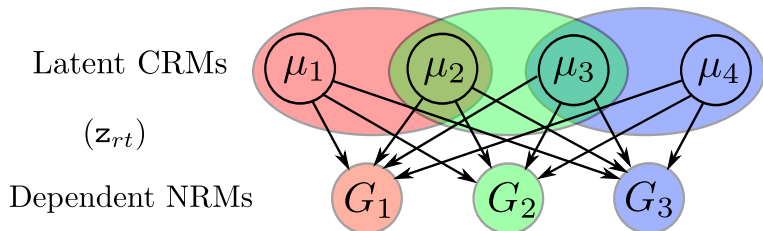
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- $\mathbf{z}\mu(\cdot)$ belongs to the same class of CRMs as $\mu(\cdot)$.
- Poisson mapping theorem:
 - ▶ If $\{w_i\}$ is a sample from a Poisson process with intensity $\nu(w)$, then $\{\mathbf{z}w_i\}$ is a Poisson process with intensity $\mathbf{z}^{-1}\nu(w/\mathbf{z})$.

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- z_{tr} governs how strongly atoms of CRM r contribute to covariate t .
- Given a set $\{z_{tr}\}$, for each t , G_t is an NRM.

Thinning theorem

- If $\{w_i\}$ is a sample from a Poisson process with intensity $\nu(w)$, then $\{z_i w_i\}$, where $z_i \stackrel{i.i.d.}{\sim} \text{Bernoulli}(p)$ is Poisson with intensity $p\nu(w)$.
- Suggests independently thinning atoms of a CRM to form a new CRM ([Lin et al., 2010]).
- Corresponds to SNRM with an exponential number of CRM.

Thinned Normalized Random Measures

- Spatial NRM characterized by a set $\{z_{tr} \in \{0, 1\} \quad \forall t \in \mathcal{T}, r \in \mathcal{R}\}$
- z_{tr} specifies whether or not *all* atoms of CRM r are present at covariate t .

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- Thinned NRM: introduce indicator variables $z_{trw} \in \{0, 1\}$ for each atom.

$$z_{rtk} \sim \text{Bernoulli}(q_{rt}) \quad k = 1, 2, \dots$$

Then, the probability measure at covariate t is given by

$$\mu_t(d\theta) = \frac{1}{\hat{\mu}_t(\Theta)} \hat{\mu}_t(d\theta), \quad \text{where } \hat{\mu}_t(d\theta) = \sum_{r=1}^R \sum_{k=1}^{\infty} z_{rtk} w_{rk} \quad (2)$$

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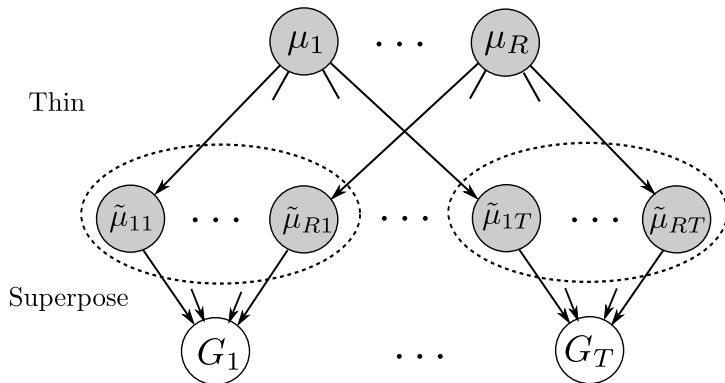
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Proposition

Conditioned on the set of q_{rt} 's, each random probability measure μ_t defined in (2) is marginally distributed as a normalized random measure with Lévy measure $\sum_r z_{rt} \nu_r(dw, d\theta)$.

Dependent normalized random measures



Inference

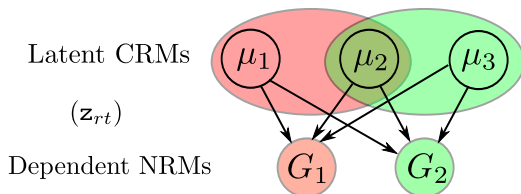
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- Unfortunately, this sampler is incorrect.
- A similar error exists in [Lin and Fisher, 2012]
- At a high level:



- ▶ One can superimpose 3 CRMs to construct 2 dependent RPMs, each marginally an NRM.
- ▶ However, given observations from one, the other is *no longer* an NRM.
- ▶ It becomes a *mixture* of NRMs.

Inference (a marginal sampling approach)

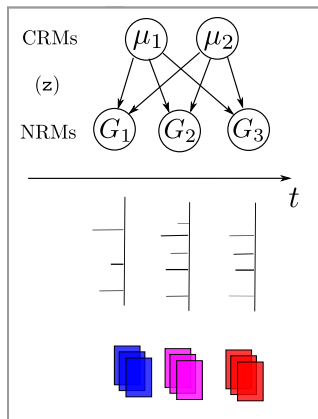
- Following [James et al., 2005], introduce auxiliary variables $u_t \forall t \in \mathcal{T}$
- Conditionally marginalize the CRMs μ_r to obtain a generalized CRP
- Alternately resample partition given $\{u_t\}$, and then $\{u_t\}$ given partition.
- Works for SNRM and MNRM, but impractical for TNRM.

Inference (a slice sampling approach)

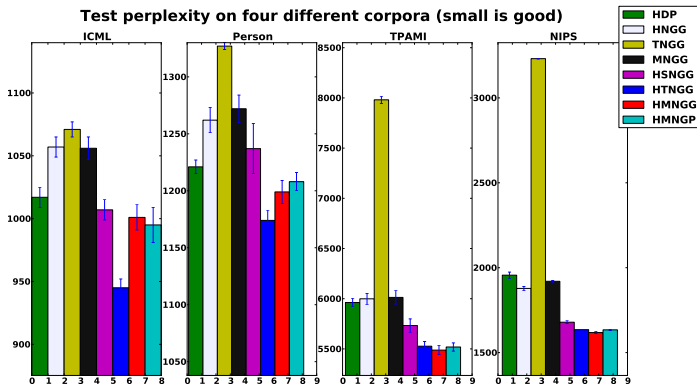
- Following [Walker, 2007], introduce auxiliary variables $s_r \forall r \in \mathcal{R}$
- Instantiate atoms of the CRM μ_r larger than s_r
- Conditionally sample partition of observations, and associated \mathbf{z}' s
- Alternately resample μ_r given $\{s_r\}$, and then $\{s_r\}$ given μ_r .

Application: Document modelling

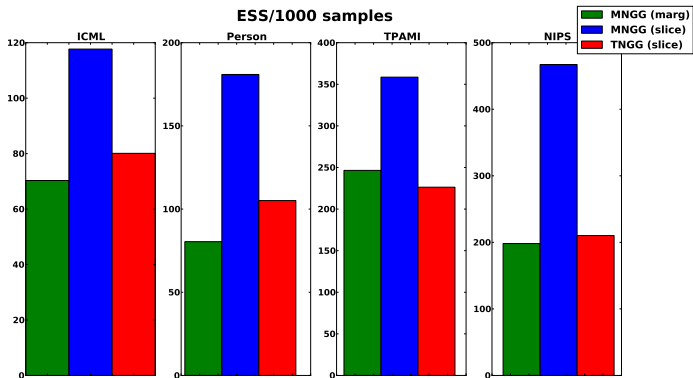
- Four corpora of documents (ICML, Person, TPAMI, NIPS)
- Documents organized by year.
- Largest corpus: NIPS
 - ▶ 17 years, 2483 documents, 3.28M words and a vocabulary of 14K
- Use a nonparametric topic model.



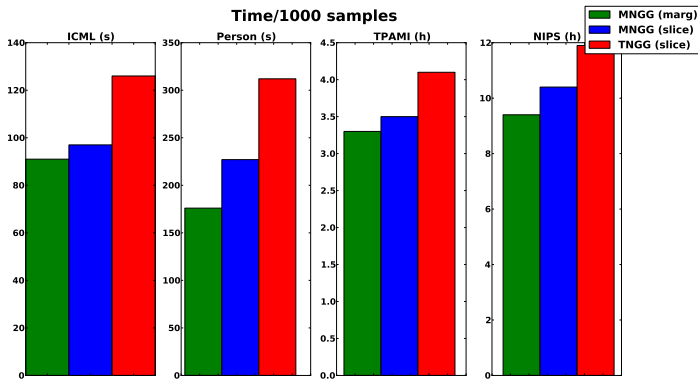
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Thank you!

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