

CSE 220: Systems Programming

Floating Point Numbers

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Floating Point

Floating point is the counterpoint to **integer** representation.

It is used to:

- represent **rational numbers**
- approximate **real numbers**

Binary floating point formats have some surprising properties.

Fixed Point

Floating point has a **closely related** representation, **fixed point**.
Fixed point is also used to represent rational and real numbers.
However, it is **less flexible** than floating point.
We will explore fixed point before floating point.

Fixed Point

A **fixed point** number has a **fixed number of digits**.

A fixed-point number has a **maximum magnitude** and **minimum fractional portion** that do not change.

For example, a fixed point number with 3 digits before and after the decimal point might include:

- 003.142
- 099.440
- 107.429

The Binary Point

In **binary numbers**, we have a **binary point**.

Just as the **decimal point** separates 10^0 from 10^{-1} , the **binary point** separates 2^0 from 2^{-1} .

Do not confuse decimal digit and decimal point!

Likewise, **binary digit and binary point**.

The Binary Point

Suppose we have a b -bit binary number with bits both **before** and **after** the binary point, such that:

- There are w whole-number bits before the binary point
- There are f fractional bits after the binary point
- The largest bit before the point is b_{w-1}
- The smallest bit before the point is b_0
- The largest bit after the point is b_{-1}
- The smallest bit after the point is b_{-f}

$$b_{w-1}, \dots, b_0, b_{-1}, \dots, b_{-f}$$

A $w.f$ -bit Binary Number

The w whole-number bits are defined as in integers:

$$b_i, i \geq 0 \doteq b_i \cdot 2^i$$

The f fractional-number bits are defined as follows:

$$b_j, j < 0 \doteq b_j \cdot 2^{-b_j}$$

Thus, its total value is:

$$\sum_{i=0}^{w-1} b_i \cdot 2^i + \sum_{j=1}^f b_j \cdot 2^{-j}$$

An Example Binary-Point Computation

Consider 11.101b:

$$\begin{aligned} 11.101b &= 1 \cdot 2^1 + 1 \cdot 2^0 + 1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3} \\ &= 2 + 1 + 1/2 + 0 + 1/8 \\ &= 3\frac{5}{8} \\ &= 3.625 \end{aligned}$$

What is “Floating Point”?

A **floating point** number, such as a **float** or **double**, is a number with a **variable number of digits before or after the decimal point**

(On computers, a variable number of **bits** before or after the **binary point!**)

Examples:

3.14159

6.022×10^{23}

6.626×10^{-34}

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Examples:

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It would take **nearly 200 bits** to represent all three of these numbers precisely.

What is “Floating Point”?

In order to represent numbers of **very small** or **very large** magnitude, floating point allows the point to **move**.

The number of **digits of precision** is fixed.

Some (loose) terms:

- **Significand**: The meaningful digits of a number
- **Exponent**: The “distance” of those digits from zero in powers of the arithmetic base

Floating Point Representation

In **base 10**, a floating point number is of the form $x \times 10^y$.

If we consider Avogadro's Number (6.022×10^{23}):

- The significand x is 6.022
- The exponent y is 23.

This requires **six digits** to store, versus 24 digits for 602200000000000000000000.

In **base 2**, a floating point number is $x \times 2^y$.

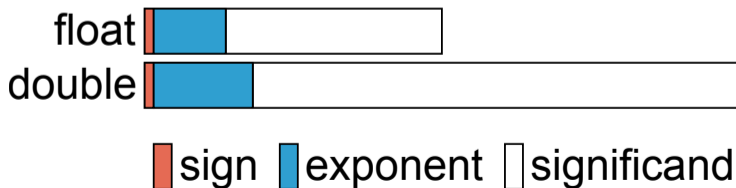
IEEE 754 Floating Point

IEEE Standard 754 defines a **particular floating point format**.

If a floating point number is $x \times 2^y$, in IEEE 754:

- A **single precision** number (**float**) has a 23-bit x and 8-bit y
- A **double precision** number (**double**) is 52-bit x and 11-bit y

Each has a one-bit **sign**.



Storing IEEE 754 Components

However, x and y are not stored directly!

x (the significand) is stored:

- Normalized to a value **right of the binary point**
- With an **assumed leading 1 preceding the binary point**

This means that a stored significand of 0 is $x = 1.0$

y (the exponent) is stored as $y + 127$.

This means that an exponent of 0 is stored as 127.

Using `dump_mem()`

We have previously used `dump_mem()` to analyze integers.

We will now use it to look at **floating point**.

Dumping a float looks like this:

```
float f = 1.0;
dump_mem(&f, sizeof(float));
```

Note that `&f` is of type `float *`, but can be passed to `void *`.

Examining Floats

```
float f1 = 2.0f;  
float f2 = 0.2f;
```

```
dump_mem(&f1, sizeof(f1));  
dump_mem(&f2, sizeof(f2));
```


Examining Floats

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```
dump_mem(&f1, sizeof(f1));  
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```

Output:

```
00 00 00 40  
cd cc 4c 3e
```

Deconstructing 2.0

Why is `2.0f 0x40000000`?

`0 10000000 00000000 00000000 00000000`

Remembering our significand and exponent storage rules, this means:

$x = 1.0$ (x is stored as significant digits **after the point**: $1 + 0$)

$y = 1$ (y is stored plus 127: $128 - 127$)

Thus: $1.0 \times 2^1 = 2.0$

(We didn't use 1.0 because it's kind of a special case.)

Deconstructing 0.2

This became 0x3e4ccccd, or:

0 01111100 1001100 11001100 11001101

Is this surprising?

Deconstructing 0.2

This became 0x3e4ccccd, or:

0 01111100 1001100 11001100 11001101

Is this surprising?

What just happened?

Deconstructing 0.2

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0 01111100 1001100 11001100 11001101

Is this surprising?

What just happened?

The significand isn't decimal!

It's after the [binary point](#).

Fractions [cleanly represented in decimal](#), like $1/5$, may not be clean in binary — sort of like $1/3$ in decimal.

More Floating Point

IEEE 754 is more complicated than we covered here.
(You’ll read more about it in the text.)

We have covered the **big ideas**, however.

Some important implications to consider:

- Very large (either positive or negative) floating point numbers **become imprecise** because of that $\times 2^y$ factor.
- Very small (close to zero) floating point numbers **become imprecise for the same reason**.
- Double precision numbers can still be quite large and precise!
- The possible floating point values are **unevenly spaced**.¹

¹See “Denormalized Values” in your text for a caveat.

Summary

- Numbers can have **fractional portions**
- Both **fixed** and **floating** point representations can be calculated in both **binary** and **decimal**
- IEEE 754 standardizes a **floating point representation**
- Floating point numbers have **fixed precision**, but **variable magnitude**

References I

Required Readings

- [1] Randal E. Bryant and David R. O'Hallaron. *Computer Science: A Programmer's Perspective*. Third Edition. Chapter 2: 2.4 Intro, 2.4.1-2.4.3, 2.4.6, 2.5. Pearson, 2016.

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