Integers and Integer Representation

CSE 220: Systems Programming

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Integers

Recall what an integer represents:
Whole numbers (positive and negative) and zero.

This is true in any numeric base.

What does 1038 mean in base 10 (decimal)?

$$1 \cdot 10^3 + 0 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0$$

Shifting left one place multiplies by the base.

Integer Complications

It seems like integers should be simple.

However, there are complications.

- Different machines use different size integers
- There are multiple possible representations
- *etc.*

In this lecture, we will explore some of these issues in C.

Effective Questions

Asking follow-up questions is also a skill.

When you get an answer, set a timer.
(Maybe 5 or 10 minutes, this time!)

Think about the answer during that time.

When the timer goes off:

- Can you make progress now?
- If not, why not?
- Do you need to ask a clarifying question?

Hexadecimal

A brief aside: we will be using hexadecimal (“hex”) a *lot*.

Hex is the base 16 numbering system.
One hex digit ranges from 0 to 15.

Contrast this to decimal, or base 10 —
one decimal digit ranges from 0 to 9.

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Contrast this to decimal, or base 10 —
one decimal digit ranges from 0 to 9.

In computing, hex digits are represented by 0-9 and then A-F.

A = 10	D = 13
B = 11	E = 14
C = 12	F = 15

Why Hex?

Hexadecimal is used because one hex digit is four bits.

This means that two hex digits represents one 8-bit byte.

On machines with 8-bit-divisible words, this is *very convenient*.

Hex	Bin	Hex	Bin
0	0000	8	1000
1	0001	9	1001
2	0010	A	1010
3	0011	B	1011
4	0100	C	1100
5	0101	D	1101
6	0110	E	1110
7	0111	F	1111

Integer Types

Platform-specific integer types you should know:

- `char`: One character.
- `short`: A short (small) integer
- `int`: An "optimally sized" integer
- `long`: A longer (bigger) integer
- `long long`: An *even longer* integer

Their sizes are: 8 bits $\leq$ `char` $\leq$ `short` $\leq$ `int` $\leq$ `long` $\leq$ `long long`

Furthermore:
`short`, `int` $\geq$ 16 bits, `long` $\geq$ 32 bits, `long long` $\geq$ 64 bits

Whew!

Integer Modifiers

Every integer type may have modifiers.

Those modifiers include `signed` and `unsigned`.

All unmodified integer types *except* `char` are signed.
`char` may be signed or unsigned!

The keyword `int` may be elided for any type except `int`.
These two declarations are equivalent:

```
long long nanoseconds;
signed long long int nanoseconds;
```

Integers of Explicit Size

The confusion of sizes has led to explicitly sized integers.

They live in `<stdint.h>`

Exact-width types are of the form `intN_t`.
They are exactly *N* bits wide; *e.g.*: `int32_t`.

Minimum-width types are of the form `int_leastN_t`.
They are at least *N* bits wide.

There are also unsigned equivalent types, which start with `u`:
`uint32_t`, `uint_least8_t`

N may be: 8, 16, 32, 64.

`dump_mem()`

In the following slides, we will use the function `dump_mem()`.

We will examine it in detail at some point, but for now:

- `dump_mem()` receives a memory address and number of bytes
- It then prints the hex values of the bytes at that address

Don't worry too much about its details for now.

A Simple Integer

First, a simple integer:

```
int x = 98303; // 0x17fff
dump_mem(&x, sizeof(x));
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Output:

```
ff 7f 01 00
```

Let's pull this apart.

Byte Ordering

Why is 98303, which is `0x17fff`, represented by `ff 7f 01 00`?

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The answer is endianness.

Words are organized into bytes in memory — but in what order?

- Big Endian: The "big end" comes first.
  This is how we write numbers.
- Little Endian: The "little end" comes first.
  This is how x86 processors (and others) represent integers.

You cannot assume anything about byte order in C!

Sign Extension

```
char c = 0x80;
int  i = c;

dump_mem(&i, sizeof(i));
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Sign Extension

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int  i = c;

dump_mem(&i, sizeof(i));
```

Output:

```
80 ff ff ff
```

`0xffffff80`? Where did all those one bits come from?!

Positive Integers

A formal definition of a positive integer on a modern machine is:

Consider an integer of width w as a vector of bits, $\vec{x}$:

$$\vec{x} = x_{w-1}, x_{w-2}, \ldots, x_0$$

This vector $\vec{x}$ has the decimal value:

$$\vec{x} \doteq \sum_{i=0}^{w-1} x_i 2^i$$

Calculating Integer Values

Consider the 8-bit binary integer `0010 1011`:

$$
\begin{aligned}
0010\,1011\text{b} &= 0 \cdot 2^7 + 0 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\
&= 0 \cdot 128 + 0 \cdot 64 + 1 \cdot 32 + 0 \cdot 16 + 1 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 1 \cdot 1 \\
&= 32 + 8 + 2 + 1 \\
&= 43
\end{aligned}
$$

Negative Integers

Previously, the variable `c` was sign extended into `i`.

As previously discussed, integers may be signed or unsigned.

Since integers are just bits, the negative numbers must have different bits set than their positive counterparts.

There are several typical ways to represent this, the most common being:

- Ones' complement
- Two's complement

Ones' Complement

Ones' complement integers represent a negative by inverting the bit pattern.

Thus, a 32-bit 1:

```
00000000 00000000 00000000 00000001
```

And a 32-bit -1:

```
11111111 11111111 11111111 11111110
```

Formally, this is like a positive integer, except:

$$x_{w-1} \doteq -2^{w-1} + 1$$

Decoding Negative Ones' Complement

Therefore, 4-bit `-1`: `1110`

$$
\begin{aligned}
1110b &= 1 \cdot (-2^3 + 1) + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\
&= 1 \cdot -7 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1 \\
&= -7 + 4 + 2 \\
&= -1
\end{aligned}
$$

Decoding Negative Ones' Complement

Therefore, 4-bit `-1`: `1110`

$$\begin{aligned} 1110b &= 1 \cdot (-2^3 + 1) + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\ &= 1 \cdot -7 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1 \\ &= -7 + 4 + 2 \\ &= -1 \end{aligned}$$

This is fine, except there are two zeroes!:

$$\begin{aligned} 0000b &= 0 \cdot (-2^3 + 1) + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 \\ 1111b &= 1 \cdot -2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= -7 + 4 + 2 + 1 \end{aligned}$$

Two's Complement

Most (modern) machines use two's complement.

Two's complement differs *slightly* from ones' complement.
Its $w - 1$th bit is defined as:

$$x_{w-1} \doteq -2^{w-1}$$

(Recall that ones' complement added 1 to this!)

This means there is only one zero — all 1s is -1!

Decoding Two's Complement

Consider 1110 in two's complement:

$$\begin{aligned} 1110b &= 1 \cdot -2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\ &= -8 + 4 + 2 + 0 \\ &= -2 \end{aligned}$$

Decoding Two's Complement

Consider 1110 in two's complement:

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&= -8 + 4 + 2 + 0 \\
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\end{aligned}
$$

w-bit Two's complement integers run from -2^{w-1} to $2^{w-1} - 1$.

Negative Integer Bit Patterns

In general, the high-order bit of a negative integer is 1.

In our previous example:

```
char c = 0x80;
int  i = c;
```

`c` is signed, and thus equivalent to -128.

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In our previous example:

```
char c = 0x80;
int  i = c;
```

`c` is signed, and thus equivalent to -128.

It is then sign extended into `i` by duplicating the high bit to the left.

This results in an `i` that also equals -128.

Why?

Computing `c` and `i`

```
char c = 0x80;
```

Here, `c` is -128 plus no other bits set.

```
int i = c;
```

What is `i` if we sign extend?

Computing c and i

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char c = 0x80;
```

Here, `c` is -128 plus no other bits set.

```c
int i = c;
```

What is `i` if we sign extend?

```
11111111 11111111 11111111 10000000
```

What is the value of that two's complement integer?

Computing Sign Extension

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```

Remember that the high 1 bit indicates -2^{w-1}, or -2^{31}, here.

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We then add in each of the other bits as positive values.

Every bit from 2^7 through 2^{30} is set, and 2^0 through 2^6 are unset:

$$-2^{31} + 2^{30} + 2^{29} + \ldots + 2^8 + 2^7$$

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…this sums to -128!

Summary

- The CPU and memory deal only in words
- Buses and registers have native word widths
- Integers have different:
  - Bit widths
  - Endianness
  - Sign representation
- Ones' and two's complement representation

Next Time …

- Scalar vs. aggregate types
- C structures
- Memory alignment

References I

Required Readings

[1] Randal E. Bryant and David R. O'Hallaron. *Computer Science: A Programmer's Perspective*. Third Edition. Chapter 2: Intro, 2.1 through 2.1.3, 2.2. Pearson, 2016.

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