

Integers and Integer Representation

CSE 220: Systems Programming

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Effective Questions

For **programming questions**, ask:

- What did I **do**?
- What did I **expect** to happen?
- What **actually** happened?
- **How are they different?**

You must know what you expected to identify the problem!

When asking us questions, **tell us** what you did, what you expected, and what you got.

Integers

Recall what an integer represents:

Whole numbers (positive and negative) and zero.

This is true in **any numeric base**.

What does 1038 mean in base 10 (decimal)?

$$1 \cdot 10^3 + 0 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0$$

Shifting left by one place **multiplies by the base**.

Integer Complications

It seems like integers should be simple.

However, **there are complications**.

- Computers are **finite**
- Different machines use **different size** integers
- There are **multiple possible representations**
- *etc.*

In this lecture, we will explore some of these issues in C.

Non-Integers

Non-integer numbers are **even more complicated**.

How do you represent a fraction, using a 1 or a 0?

Different bases express different **rational numbers**.

Real numbers are infinite, but computers are finite.

We will only touch on non-integers this semester.

Hexadecimal

A brief aside: we will be using **hexadecimal** (“hex”) a *lot*.

Hex is the **base 16** numbering system.

One hex digit ranges from 0 to 15.

Contrast this to **decimal**, or **base 10** —
one decimal digit ranges from 0 to 9.

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In computing, hex digits are represented by 0-9 and then **A-F**.

A = 10 D = 13

B = 11 E = 14

C = 12 F = 15

Why Hex?

Hexadecimal is used because **one hex digit is four bits**.

This means that **two hex digits** represents **one 8-bit byte**.

On machines with 8-bit-divisible words, this is *very convenient*.

Hex	Bin	Hex	Bin
0	0000	8	1000
1	0001	9	1001
2	0010	A	1010
3	0011	B	1011
4	0100	C	1100
5	0101	D	1101
6	0110	E	1110
7	0111	F	1111

Integer Types

Platform-specific integer types **you should know**:

- `char`: One character.
- `short`: A short (small) integer
- `int`: An “optimally sized” integer
- `long`: A longer (bigger) integer
- `long long`: An *even longer* integer

Their sizes are: $8 \text{ bits} \leq \text{char} \leq \text{short} \leq \text{int} \leq \text{long} \leq \text{long long}$

Furthermore:

`short`, `int` ≥ 16 bits, `long` ≥ 32 bits, `long long` ≥ 64 bits

Whew!

Integer Modifiers

Every integer type may have **modifiers**.

Those modifiers include **signed** and **unsigned**.

All unmodified integer types *except* **char** are **signed**.
char may be signed or unsigned!

The keyword **int** may be elided for any type except **int**.
These two declarations are equivalent:

```
long long nanoseconds;  
signed long long int nanoseconds;
```

Integers of Explicit Size

The **confusion of sizes** has led to **explicitly sized** integers.

They live in `<stdint.h>`

Exact-width types are of the form `intN_t`.

They are exactly ***N* bits wide**; e.g.: `int32_t`.

Minimum-width types are of the form `int_leastN_t`.

They are **at least *N* bits wide**.

There are also **unsigned** equivalent types, which start with `u`:
`uint32_t`, `uint_least8_t`

N may be: 8, 16, 32, 64.

dump_mem()

In the following slides, we will use the function `dump_mem()`.

We will examine it in detail at some point, but for now:

- `dump_mem()` receives a **memory address** and **number of bytes**
- It then **prints the hex values** of the bytes at that address

Don't worry too much about its details for now.

A Simple Integer

First, a simple integer:

```
int x = 98303; // hex 0x17fff
dump_mem(&x, sizeof(x));
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Output:

```
ff 7f 01 00
```

Let's pull this apart.

Byte Ordering

Why is 98303, which is $0x17fff$, represented by `ff 7f 01 00`?

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The answer is **endianness**.

Words are organized into **bytes** in memory — but in what order?

- **Big Endian**: The “big end” comes first.
This is how we **write numbers**.
- **Little Endian**: The “little end” comes first.
This is how x86 processors (and others) represent integers.

You **cannot assume anything about byte order** in C!

Sign Extension

```
char c = 0x80;  
int i = c;  
  
dump_mem(&i, sizeof(i));
```

Sign Extension

```
char c = 0x80;  
int i = c;  
  
dump_mem(&i, sizeof(i));
```

Output:

80 ff ff ff

0xffffffff80? Where did all those one bits come from?!

Positive Integers

A formal definition of a positive integer on a modern machine is:

Consider an integer of width w as a vector of bits, \vec{x} :

$$\vec{x} = x_{w-1}, x_{w-2}, \dots, x_0$$

This vector \vec{x} has the **decimal value**:

$$\vec{x} \doteq \sum_{i=0}^{w-1} x_i 2^i$$

Calculating Integer Values

Consider the 8-bit binary integer 0010 1011:

$$\begin{aligned} 0010\ 1011\mathbf{b} &= 0 \cdot 2^7 + 0 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 0 \cdot 128 + 0 \cdot 64 + 1 \cdot 32 + 0 \cdot 16 + 1 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 1 \cdot 1 \\ &= 32 + 8 + 2 + 1 \\ &= 43 \end{aligned}$$

Negative Integers

Previously, the variable `c` was **sign extended** into `i`.

As previously discussed, integers may be **signed** or **unsigned**.

Since **integers are just bits**, the **negative numbers** must have **different bits set** than their positive counterparts.

There are several typical ways to represent this, the most common being:

- Ones' complement
- Two's complement

Ones' Complement

Ones' complement integers represent a negative by **inverting the bit pattern**.

Thus, a 32-bit 1:

00000000 00000000 00000000 00000001

And a 32-bit -1:

11111111 11111111 11111111 11111110

Formally, this is **like a positive integer**, except:

$$x_{w-1} \doteq -2^{w-1} + 1$$

Decoding Negative Ones' Complement

Therefore, 4-bit -1: 1110

$$\begin{aligned} 1110_{\text{b}} &= 1 \cdot (-2^3 + 1) + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\ &= 1 \cdot -7 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1 \\ &= -7 + 4 + 2 \\ &= -1 \end{aligned}$$

Decoding Negative Ones' Complement

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This is fine, except **there are two zeroes!**:

$$\begin{aligned}0000\text{b} &= 0 \cdot (-2^3 + 1) + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 \\ 1111\text{b} &= 1 \cdot -2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= -7 + 4 + 2 + 1\end{aligned}$$

Two's Complement

Most (modern) machines use **two's complement**.

Two's complement differs *slightly* from ones' complement.
Its $w - 1$ th bit is defined as:

$$x_{w-1} \doteq -2^{w-1}$$

(Recall that ones' complement added 1 to this!)

This means there is **only one zero** — all 1s is -1!

Decoding Two's Complement

Consider 1110 in two's complement:

$$\begin{aligned} 1110\text{b} &= 1 \cdot -2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\ &= -8 + 4 + 2 + 0 \\ &= -2 \end{aligned}$$

Decoding Two's Complement

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w -bit Two's complement integers run from -2^{w-1} to $2^{w-1} - 1$.

Negative Integer Bit Patterns

In general, the high-order bit of a negative integer is 1.

In our previous example:

```
char c = 0x80;  
int  i = c;
```

c is **signed**, and thus equivalent to -128.

Negative Integer Bit Patterns

In general, the high-order bit of a negative integer is 1.

In our previous example:

```
char c = 0x80;  
int  i = c;
```

c is **signed**, and thus equivalent to -128.

It is then **sign extended** into i by **duplicating the high bit to the left**.

This results in an i that **also equals -128**.

Why?

Computing c and i

```
char c = 0x80;
```

Here, c is -128 plus **no other bits set**.

```
int i = c;
```

What is i if we sign extend?

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char c = 0x80;
```

Here, c is -128 plus **no other bits set**.

```
int i = c;
```

What is i if we sign extend?

```
11111111 11111111 11111111 10000000
```

What is the value of that two's complement integer?

Computing Sign Extension

11111111 11111111 11111111 10000000

Remember that the high 1 bit indicates -2^{w-1} , or -2^{31} , here.

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We then add in each of the other bits as **positive** values.

Every bit from 2^7 through 2^{30} is set, and 2^0 through 2^6 are unset:

$$-2^{31} + 2^{30} + 2^{29} + \dots + 2^8 + 2^7$$

Computing Sign Extension

11111111 11111111 11111111 10000000

Remember that the high 1 bit indicates -2^{w-1} , or -2^{31} , here.

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Every bit from 2^7 through 2^{30} is set, and 2^0 through 2^6 are unset:

$$-2^{31} + 2^{30} + 2^{29} + \dots + 2^8 + 2^7$$

...this sums to -128!

Representing Fractional Values

What if we want to represent **non-integers**?

We can assign certain bits to 2^{-1} , 2^{-2} , *etc.*

This is called **fixed point**.

Fixed point assigns a **specific number** of bits to:

- fractions
- whole numbers

This works well for numbers of **moderate size and precision**.

Floating Point

What if you want **more range**?

You can **move the (binary) point**, like scientific notation:

$$x \times 2^y$$

... but **how do you encode** the point?

There is no . in 0 or 1!

We use special patterns of bits called **floating point**.¹

You'll learn more in CSE 341.

¹Remember that there's also no -.

Summary

- The CPU and memory deal **only in words**
- Buses and registers have **native word widths**
- Integers have different:
 - Bit widths
 - **Endianness**
 - Sign representation
- **Ones' and two's complement** representation
- Bits also have to represent **fractional values**.

Next Time ...

- Scalar vs. aggregate types
- C structures
- Memory alignment

References I

Required Readings

- [1] Randal E. Bryant and David R. O'Hallaron. *Computer Science: A Programmer's Perspective*. Third Edition. Chapter 2: Intro, 2.1 through 2.1.3, 2.2. Pearson, 2016.

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