Integers and Integer Representation

CSE 220: Systems Programming

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Recall what an integer represents:
Whole numbers (positive and negative) and zero.

This is true in any numeric base.

What does 1038 mean in base 10 (decimal)?

\[ 1 \cdot 10^3 + 0 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0 \]

Shifting left by one place multiplies by the base.
Integer Complications

It seems like integers should be simple.

However, there are complications.

- Computers are finite
- Different machines use different size integers
- There are multiple possible representations
- etc.

In this lecture, we will explore some of these issues in C.
Non-Integers

Non-integer numbers are even more complicated.

How do you represent a fraction, using a 1 or a 0?

Different bases express different rational numbers.

Real numbers are infinite, but computers are finite.

We will only touch on non-integers this semester.
Hexadecimal

A brief aside: we will be using hexadecimal ("hex") a lot.

Hex is the base 16 numbering system. One hex digit ranges from 0 to 15.

Contrast this to decimal, or base 10 — one decimal digit ranges from 0 to 9.
Hexadecimal

A brief aside: we will be using hexadecimal ("hex") a lot.

Hex is the base 16 numbering system. One hex digit ranges from 0 to 15.

Contrast this to decimal, or base 10 — one decimal digit ranges from 0 to 9.

In computing, hex digits are represented by 0-9 and then A-F.

A = 10  D = 13
B = 11  E = 14
C = 12  F = 15
Why Hex?

Hexadecimal is used because one hex digit is four bits.

This means that two hex digits represents one 8-bit byte.

On machines with 8-bit-divisible words, this is very convenient.

<table>
<thead>
<tr>
<th>Hex</th>
<th>Bin</th>
<th>Hex</th>
<th>Bin</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0000</td>
<td>8</td>
<td>1000</td>
</tr>
<tr>
<td>1</td>
<td>0001</td>
<td>9</td>
<td>1001</td>
</tr>
<tr>
<td>2</td>
<td>0010</td>
<td>A</td>
<td>1010</td>
</tr>
<tr>
<td>3</td>
<td>0011</td>
<td>B</td>
<td>1011</td>
</tr>
<tr>
<td>4</td>
<td>0100</td>
<td>C</td>
<td>1100</td>
</tr>
<tr>
<td>5</td>
<td>0101</td>
<td>D</td>
<td>1101</td>
</tr>
<tr>
<td>6</td>
<td>0110</td>
<td>E</td>
<td>1110</td>
</tr>
<tr>
<td>7</td>
<td>0111</td>
<td>F</td>
<td>1111</td>
</tr>
</tbody>
</table>
Integer Types

Platform-specific integer types you should know:

- **char**: One character.
- **short**: A short (small) integer
- **int**: An “optimally sized” integer
- **long**: A longer (bigger) integer
- **long long**: An *even longer* integer

Their sizes are: 8 bits $\leq$ char $\leq$ short $\leq$ int $\leq$ long $\leq$ long long

Furthermore:

short, int $\geq$ 16 bits, long $\geq$ 32 bits, long long $\geq$ 64 bits

*Whew!*
Every integer type may have modifiers.

Those modifiers include signed and unsigned.

All unmodified integer types except char are signed. char may be signed or unsigned!

The keyword int may be elided for any type except int. These two declarations are equivalent:

```c
long long nanoseconds;
signed long long int nanoseCONDS;
```
Integers of Explicit Size

The confusion of sizes has led to explicitly sized integers. They live in `<stdint.h>`.

**Exact-width** types are of the form `intN_t`. They are exactly $N$ bits wide; e.g.: `int32_t`.

**Minimum-width** types are of the form `int_leastN_t`. They are at least $N$ bits wide.

There are also **unsigned** equivalent types, which start with `u`: `uint32_t`, `uint_least8_t`

$N$ may be: 8, 16, 32, 64.
dump_mem()

In the following slides, we will use the function `dump_mem()`.

We will examine it in detail at some point, but for now:

- `dump_mem()` receives a memory address and number of bytes
- It then prints the hex values of the bytes at that address

Don’t worry too much about its details for now.
A Simple Integer

First, a simple integer:

```c
int x = 98303; // hex 0x17fff
dump_mem(&x, sizeof(x));
```
A Simple Integer

First, a simple integer:

```c
int x = 98303; // hex 0x17fff
dump_mem(&x, sizeof(x));
```

Output:
```
ff 7f 01 00
```

Let’s pull this apart.
Byte Ordering

Why is 98303, which is 0x17fff, represented by ff 7f 01 00?
Why is 98303, which is 0x17fff, represented by ff 7f 01 00?

The answer is **endianness**.

Words are organized into bytes in memory — but in what order?

- **Big Endian**: The “big end” comes first. This is how we **write numbers**.
- **Little Endian**: The “little end” comes first. This is how x86 processors (and others) represent integers.

You **cannot assume anything about byte order** in C!
Sign Extension

```c
char c = 0x80;
int i = c;

dump_mem(&i, sizeof(i));
```
Sign Extension

```c
char c = 0x80;
int i = c;

dump_mem(&i, sizeof(i));
```

Output:
80 ff ff ff

0xffffffff80? Where did all those one bits come from?!
Positive Integers

A formal definition of a positive integer on a modern machine is:

Consider an integer of width $w$ as a vector of bits, $\vec{x}$:

$$\vec{x} = x_{w-1}, x_{w-2}, \ldots, x_0$$

This vector $\vec{x}$ has the decimal value:

$$\vec{x} = \sum_{i=0}^{w-1} x_i 2^i$$
Calculating Integer Values

Consider the 8-bit binary integer \(0010\ 1011\):

\[
\begin{align*}
0010\ 1011_b &= 0 \cdot 2^7 + 0 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\
&= 0 \cdot 128 + 0 \cdot 64 + 1 \cdot 32 + 0 \cdot 16 + 1 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 1 \cdot 1 \\
&= 32 + 8 + 2 + 1 \\
&= 43
\end{align*}
\]
Negative Integers

Previously, the variable \( c \) was sign extended into \( i \).

As previously discussed, integers may be signed or unsigned.

Since integers are just bits, the negative numbers must have different bits set than their positive counterparts.

There are several typical ways to represent this, the most common being:

- Ones’ complement
- Two’s complement
Ones’ Complement

Ones’ complement integers represent a negative by inverting the bit pattern.

Thus, a 32-bit 1:
00000000 00000000 00000000 00000001

And a 32-bit -1:
11111111 11111111 11111111 11111110

Formally, this is like a positive integer, except:

\[ x_{w-1} \doteq -2^{w-1} + 1 \]
Decoding Negative Ones’ Complement

Therefore, 4-bit -1: 1110

\[
\begin{align*}
1110_b &= 1 \cdot (-2^3 + 1) + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\
&= 1 \cdot -7 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1 \\
&= -7 + 4 + 2 \\
&= -1
\end{align*}
\]
Decoding Negative Ones’ Complement

Therefore, 4-bit $-1$: $1110$

$$1110b = 1 \cdot (-2^3 + 1) + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$$
$$= 1 \cdot -7 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1$$
$$= -7 + 4 + 2$$
$$= -1$$

This is fine, except there are two zeroes!:

$$0000b = 0 \cdot (-2^3 + 1) + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$$

$$1111b = 1 \cdot -2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$$
$$= -7 + 4 + 2 + 1$$
Two’s Complement

Most (modern) machines use two’s complement.

Two’s complement differs *slightly* from ones’ complement. Its $w - 1$th bit is defined as:

$$x_{w-1} = -2^{w-1}$$

(Recall that ones’ complement added 1 to this!)

This means there is only one zero — all 1s is -1!
4-bit wide two's complement

- Non-negative numbers:
  - 0000
  - 0001
  - 0010
  - 0011
  - 0100
  - 0110
  - 0101
  - 0111

- Negative numbers:
  - 1111
  - 1010
  - 1110
  - 1011
  - 1101
  - 1100
  - 1000
  - 1001
  - 1010
  - 1011

Arrow shows the effect of negation.
Decoding Two’s Complement

Consider 1110 in two’s complement:

\[
1110_{\text{b}} = 1 \cdot -2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\
= -8 + 4 + 2 + 0 \\
= -2
\]
Decoding Two’s Complement

Consider 1110 in two’s complement:

\[
1110_b = 1 \cdot -2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\
= -8 + 4 + 2 + 0 \\
= -2
\]

\(w\)-bit Two’s complement integers run from \(-2^{w-1}\) to \(2^{w-1} - 1\).
Negative Integer Bit Patterns

In general, the high-order bit of a negative integer is 1.

In our previous example:

```c
char c = 0x80;
int i = c;
```

c is signed, and thus equivalent to -128.
Negative Integer Bit Patterns

In general, the high-order bit of a negative integer is 1.

In our previous example:

```c
char c = 0x80;
int i = c;
```

c is signed, and thus equivalent to -128.

It is then sign extended into i by duplicating the high bit to the left.

This results in an i that also equals -128.

Why?
Computing c and i

```cpp
char c = 0x80;
Here, c is -128 plus no other bits set.
int i = c;
What is i if we sign extend?
```
Computing c and i

```c
char c = 0x80;
Here, c is -128 plus no other bits set.

int i = c;
What is i if we sign extend?

11111111 11111111 11111111 10000000

What is the value of that two’s complement integer?
Computing Sign Extension

11111111 11111111 11111111 10000000

Remember that the high 1 bit indicates $-2^{w-1}$, or $-2^{31}$, here.
Computing Sign Extension

11111111 11111111 11111111 10000000

Remember that the high 1 bit indicates $-2^{w-1}$, or $-2^{31}$, here.

We then add in each of the other bits as positive values.

Every bit from $2^7$ through $2^{30}$ is set, and $2^0$ through $2^6$ are unset:

$$-2^{31} + 2^{30} + 2^{29} + \ldots + 2^8 + 2^7$$
Computing Sign Extension

11111111 11111111 11111111 10000000

Remember that the high 1 bit indicates \(-2^{w-1}\), or \(-2^{31}\), here.

We then add in each of the other bits as positive values.

Every bit from \(2^7\) through \(2^{30}\) is set, and \(2^0\) through \(2^6\) are unset:

\[-2^{31} + 2^{30} + 2^{29} + \ldots + 2^8 + 2^7\]

…this sums to -128!
Representing Fractional Values

What if we want to represent non-integers?

We can assign certain bits to $2^{-1}$, $2^{-2}$, etc.

This is called fixed point.

Fixed point assigns a specific number of bits to:

- fractions
- whole numbers

This works well for numbers of moderate size and precision.
Floating Point

What if you want more range?

You can move the (binary) point, like scientific notation:

\[ x \times 2^y \]

... but how do you encode the point?

There is no . in 0 or 1!

We use special patterns of bits called floating point.¹

You’ll learn more in CSE 341.

¹Remember that there’s also no -. 
The CPU and memory deal only in words

Buses and registers have native word widths

Integers have different:
- Bit widths
- Endianness
- Sign representation

Ones’ and two’s complement representation

Bits also have to represent fractional values.
Next Time …

- Scalar vs. aggregate types
- C structures
- Memory alignment
References

Required Readings


Optional Readings

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