Homework #4

Due: 4/23/23 @ 11:59pm

Content Covered: Relations, Functions, Sequences, Recurrence Relations, Induction

Submission Instructions

Submit your completed homework to UBLearns electronically in PDF format. Any submissions that are not a PDF or not a legible PDF will not receive credit. We need to be able to read your submission to be able to grade your work. Your write-up should contain enough information from the problem so that a reader doesn't need to return to the text to know what the problem is (it is a good habit to rewrite each problem prior to solving it). There is no general rule for how much information from the problem to include, but it should be possible to read your homework and ascertain what the problem was and what your solution is accomplishing.

When writing up the solution, you may hand write the solutions and submit a scanned PDF, or write up the solutions electronically and convert them to a PDF. If you hand write your solutions, make sure that you write clearly and your writing is legible. Double check your scans to make sure that your scanned copy is legible. After you submit your work, make sure the file is visible. Download your submitted copy, open it, and see whether you submitted the correct file and your submitted file has not been corrupted during the upload.

You are able to upload your submission multiple times. Only the last file will be graded. Keep in mind that if your completed work consists of multiple pages and you submit a separate file for each page, only the last file submitted will be graded. In this case, only one page of your submission would be graded. **You are responsible for making sure that your submission goes through as intended.**

Your submitted work must be your own. Please review the course Academic Integrity Policy as outlined in the syllabus. **Failure to adhere to this policy will result in an F in the course.**

Late Policy

Late homework will be accepted up to 1 day late for a penalty of 25% of the total points. For example, if the homework is worth 100 points and you submit it one day late, you will receive the maximum of (your score earned minus 25 points) and 0 points.

Please be mindful of the deadlines, and start assignments early. Course staff will likely be less available after 5PM and during weekends, so plan accordingly if you need assistance.

Problems [50 points]

Problem 1 [10 points]

Consider the set ***S*** = {A, B, C, D}:

For each of the following relations on ***S***, determine whether it is a partial order, a total order, an equivalence relation, or none of the above, and in one sentence explain why:

RUBRIC: 1 point for correct answer, 1 point for valid explanation

a) {(A,B), (B,C), (C,D), (A,C), (B,D), (A,D)}
   None of the above: it is not reflexive
b) {(A,A), (B,A), (B,B), (C,C), (C,D), (D,D)}
   Partial order: it is reflexive, anti-symmetric, transitive, but not defined for all pairs
c) {(A,A), (B,A), (A,B), (B,B), (C,C), (A,C), (C,A), (D,D)}
   None of the above: it is not transitive
d) {(A,A), (B,A), (A,B), (B,B), (C,C), (D,D)}
   Equivalence relation: it is reflexive, symmetric, and transitive
e) {(A,A), (B,A), (A,B), (B,B), (C,C), (C,D), (D,D)}
   None of the above: it is not symmetric or anti-symmetric

Problem 2 [6 points]

For each of the following functions, determine whether it is injective, surjective, and/or invertible and explain why:

RUBRIC: 1 point for correct answer, 1 point for valid explanation

a) Function f: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f((a, b)) = b + a$.
   Surjective: For any z in z, (z,0) maps to z.
b) Let ***A* = {1, 2, 4}** and function **sum:** $\mathscr{P}(A) \to$ **{0, 1, 2, 3, 4, 5, 6, 7}**, where **sum**(***X***) maps to the sum of all elements of ***X***. **Reminder:** $\mathscr{P}(A)$ is the powerset of ***A***.
   Injective, Surjective, Invertible: There are 8 elements in P(A) and each one maps to a unique element of {0…7}.
c) Function f: $\mathbb{R} \to \mathbb{R}$ is defined as $f(x) = (x^2 + 1)/(x^2 + 2)$
   None of the above. +1 and -1 map to the same value, and nothing maps to negative numbers.

Problem 3 [4 points]

Let set ***G* = { Monopoly, Catan, Deus, Orleans, Imhotep, Splendor, Clue }** and let the relation ***R***: ***G*** → ***G*** be defined such that ***a R b*** iff ***a* = *b*** or the title of ***a*** is shorter than the title of ***b***. For example, (**Clue**, **Catan**) is in ***R*** because **"Clue"** is shorter than **"Catan"**. **(*G, R*)** is a poset.

Draw the Hasse diagram for this partial ordering.

RUBRIC: Deduct 1 point for each missing/extra line, or if upside-down. See last page.

Problem 4 [10 points]

For each sequence below, (i) write out the first 5 terms, (ii) write out the explicit formula for the term with index n, and (iii) describe the sequence using a recurrence relation.
RUBRIC: 1 point for first 5 terms, 2 points for explicit formula, 1 point for recurrence initial condition, 1 point for recurrence relation

a) **{a_n}** is the geometric sequence with initial term **a_0 = -16** and common ratio ¼
   (i) -16, -4, -1, -¼, -1/16
   (ii) a_n = -16 * $(¼)^n$
   (iii) a_0 = -16, a_n = a_{n-1} * ¼

b) **{b_n}** is the arithmetic sequence with initial term **b_0 = 1999** and common difference 4
   (i) 1999, 2003, 2007, 2011, 2015
   (ii) a_n = 1999 + 4n
   (iii) a_0 = 1999, a_n = a_{n-1} + 4

Problem 5 [10 points]

For each of the below recurrence relations, solve to find an explicit formula by using iteration. Show all work.
RUBRIC: 2 points for correct explicit formula. 3 points for showing reasonable work.

a) $a_0 = 0$, $a_n = a_{n-1} + 2n$
   Solution: a_n = n * (n + 1)
   Example Work:
   a_0 = 0
   a_1 = 0 + 2 * 1 = 2
   a_2 = (0 + 2 * 1) + 2 * 2 = 6
   a_3 = ((0 + 2 * 1) + 2 * 2) + 2 * 3 = 12
   a_4 = (((0 + 2 * 1) + 2 * 2) + 2 * 3) + 2 * 4 = 20
   a_n = (0 + 1 + 2 + 3 + 4 + … + n) * 2 = n * (n+1) / 2 * 2 = n * (n+1)

b) $a_0 = 4$, $a_n = ½\, a_{n-1} + 2$
   Solution: a_n = 4
   Example Work:
   a_0 = 4
   a_1 = 4/2 + 2 = 4
   a_2 = (4/2 + 2) / 2 + 2 = 4
   a_3 = ((4/2 + 2) / 2 + 2)/2 + 2 = 4

Problem 6 [10 points]

Let $\{a_n\}$ be the sequence defined by $a_0 = 0$, $a_1 = 3$, and $a_n = 5a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Prove by strong mathematical induction that $a_n = 4^n - 1$ for all $n \geq 0$.

Your proof must include every part of the proof template referenced in class

RUBRIC:

2 points for step 1 (deduct a point per minor mistake, ie n ≥ some other number)

2 points for step 2 (deduct a point per minor mistake)

4 points for step 3 (deduct a point per minor mistake)

2 points for step 4 (deduct a point per minor mistake)

1. **Let P(n): $a_n = 4^n - 1$. We want to show that P(n) is true for all n ≥ 0.**
2. **Base Step:**
   P(0): $a_0 = 4^0 - 1$. $a_0 = 0$, $4^0 - 1 = 0$.
   P(1): $a_1 = 4^1 - 1$. $a_1 = 3$, $4^1 - 1 = 3$.
3. **Inductive Step:**
   a. **Assume that P(i) is true for all 0 ≤ i ≤ k, for any k ≥ 2**
   b. **We will now show that P(k+1) must be true P(k+1): $a_{k+1} = 4^{k+1} - 1$**
   c. **$a_{k+1} = 5a_k - 4a_{k-1}$ (by definition)**
      By our assumption, $a_k = 4^k$-1 and $a_{k-1} = 4^{k-1}$-1.
      Therefore $a_{k+1} = 5(4^k-1) - 4(4^{k-1}-1) = 5(4^k-1) - 4^k + 4 = 5*4^k - 5 - 4^k + 4 = 4*4^k - 1 = 4^{k+1} - 1$
   d. **Therefore by our inductive assumption, P(i) for all 0 ≤ i ≤ k, and any k ≥ 2 implies P(k+1)**
4. **Therefore by the principle of strong mathematical induction, P(n) is true for all n≥0**

Problem 7 (Extra Credit) [5 points]

Answer each of the following:

RUBRIC: 1 point each if completely correct

a) $\lfloor -2.1 \rfloor$ = ?
   -3

b) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$ = ?
   1

c) $\lfloor a \rfloor = \lceil a \rceil$ for all a ∈ ℝ. True or False? Give a counterexample if false.
   False. $\lfloor 1/2 \rfloor = 0$ $\lceil 1/2 \rceil = 1$

d) $\lfloor a \rfloor = \lceil a \rceil - 1 = \lceil a - 1 \rceil$ for all a ∈ R – ℤ. True or False? Give a counterexample if false.
   True

e) Is 40 congruent to 5 modulo 17? Why, or why not?
   No. (40 - 5) mod 17 = 1

ANSWER FOR Q3