Homework #5

Due: 5/10/23 @ 11:59pm

Content Covered: Counting, Graphs

Submission Instructions

Submit your completed homework to UBLearns electronically in PDF format. Any submissions that are not a PDF or not a legible PDF will not receive credit. We need to be able to read your submission to be able to grade your work. Your write-up should contain enough information from the problem so that a reader doesn't need to return to the text to know what the problem is (it is a good habit to rewrite each problem prior to solving it). There is no general rule for how much information from the problem to include, but it should be possible to read your homework and ascertain what the problem was and what your solution is accomplishing.

When writing up the solution, you may hand write the solutions and submit a scanned PDF, or write up the solutions electronically and convert them to a PDF. If you hand write your solutions, make sure that you write clearly and your writing is legible. Double check your scans to make sure that your scanned copy is legible. After you submit your work, make sure the file is visible. Download your submitted copy, open it, and see whether you submitted the correct file and your submitted file has not been corrupted during the upload.

You are able to upload your submission multiple times. Only the last file will be graded. Keep in mind that if your completed work consists of multiple pages and you submit a separate file for each page, only the last file submitted will be graded. In this case, only one page of your submission would be graded. **You are responsible for making sure that your submission goes through as intended.**

Your submitted work must be your own. Please review the course Academic Integrity Policy as outlined in the syllabus. **Failure to adhere to this policy will result in an F in the course.**

Late Policy

Late homework will be accepted up to 1 day late for a penalty of 25% of the total points. For example, if the homework is worth 100 points and you submit it one day late, you will receive the maximum of (your score earned minus 25 points) and 0 points.

Please be mindful of the deadlines, and start assignments early. Course staff will likely be less available after 5PM and during weekends, so plan accordingly if you need assistance.

Problems [50 points]

Problem 1 [15 points]

Answer each of the following counting problems. For each problem you must explain how you got your answer. **Answers that show no work get no credit, even if the number is correct.**

Valid explanations should include things like which counting rules were applied (ie sum rule, product rule), and why order does or does not matter, when relevant, etc. For permutations and combinations, leave your answers in terms of $P(n, k)$ and $C(n, k)$.
RUBRIC: 1 point for correct answer (if work is shown), 2 points for work/justification. Give credit for valid work/justification if answer is close (ie off by one)

a) Alice owns 3 dresses, 6 pairs of casual pants, 4 casual shirts, 2 pairs of fancy pants, and 3 fancy shirts. If she does not mix fancy and casual, how many outfits can she make that include either a dress, or shirt and pants?

3 + 6*4 + 3*2. Each shirt can be pair with multiple pairs of pants, so the product rule is used for shirt/pants combos. But Alice will only wear one outfit that is either dress OR fancy OR casual, so we need sum rule as well.

b) How many possible poker hands can you make from a standard deck of playing cards? (A poker hand is a selection of 5 cards from a deck of 52 unique cards)

C(52, 5) Order doesn't matter. We are selecting a combination of 5 from 52.

c) A sequence of DNA can be represented as a sequence of bases, where the possible bases are A, C, G, T. How many DNA sequences of length 5 that do not contain the sequence TAG exist?

$4^5 - (4^2 * 3)$. The total number of 5 length involves choosing from the 4 letters 5 times in a row. So use product rule to get 4^5. Then we need to subtract number of sequences that contain TAG. Treat TAG as one letter. It can be first (followed by 2 letters), second (between 2), or third (following 2). We have 4^2 ways to choose the 2 letters (product rule), and then add together the 3 options (sum rule).

d) A collectible trading card game sells packs that contain 10 random cards each. If there are 150 possible cards, how many packs must you buy to **guarantee** you get at least 4 of the same card?

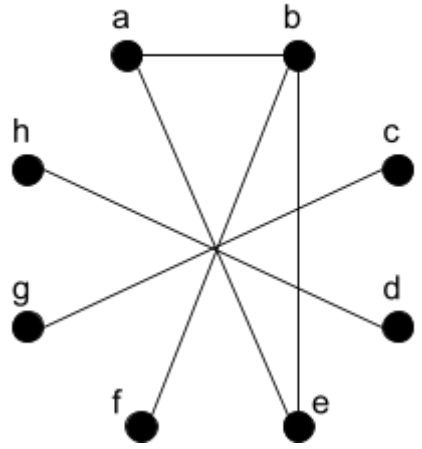
Need 451 cards, so we need to buy 46 packs. By pigeonhole principle, since 451/150 = 3.006… then we must have at least one card that we have at least 4 of.

e) Suppose we have everyone in the room rank their top 5 Star Wars movies in order from favorite to least favorite (there are 9 Star Wars movies to choose from). How many people must be in the room to **guarantee** that two people in the room have the same exact ranking?

P(9,5) + 1. There are P(9,5) ways to pick an ordered top five, so adding one more person means that two people in the room have to match (by pigeonhole principle)

Problem 2 [10 points]

Let $G_1 = (V_1, E_1)$ be the following graph:



a) Write out E_1 in roster notation.
{{a,b},{a,e},{b,f},{b,e},{c,g},{d,h}}

b) What is the total degree of G_1?
12

c) Write out the connected components of G_1.
{a,b,e,f},{c,g},{d,h}

d) Do the connected components partition V_1?
Yes

e) Write out the neighborhood of **{e,f}**.
{a,b}

Problem 3 [10 points]

Let $G_2 = (V_2, E_2)$ be the directed graph represented by the following adjacency matrix:

	1	2	3	4
1	0	0	1	1
2	1	1	0	0
3	0	1	0	0
4	0	1	0	0

a) Draw G_2. **RUBRIC: 2 points if correct, 1 if off by just one edge or flipped/no arrows.**
See last page

b) What is $|V_2|$? What is $|E_2|$? **RUBRIC: 1 point for each answer.**
4 and 6

c) Is G_2 a simple graph? Why or why not? **RUBRIC: 1 point answer, 1 point justification**
No. 2 has a loop

d) Which vertex has the largest out-degree? The largest in-degree? **RUBRIC: 1 point each**
1 or 2 and 2

e) A directed graph is weakly connected if replacing every directed edge with an undirected edge results in a connected graph. A directed graph is strongly connected if for every pair of vertices (x,y) there is a directed path from x to y (and vice-versa). Is G_2 strongly connected, weakly connected, or neither? **RUBRIC: 2 points if correct.**
Strongly connected

Problem 4 [6 points]

A degree sequence of a graph is the sequence of degrees of the vertices of the graph in non-increasing order. For example, the degree sequence of graph G_1 from Problem 1 is 3, 2, 2, 1, 1, 1, 1, 1. An arbitrary sequence is **graphic** if it can be the degree sequence of a simple graph. For each of the following sequences, state whether they are graphic or not. If they are not graphic, state why. If they are graphic, draw a **connected** graph with that degree sequence.
RUBRIC: 1 point for answer, 1 point for correct drawing or justification

a) 5, 3, 2, 2, 2, 2 **Yes. Graph on last page.**

b) 4, 3, 2, 1, 1 **No. Sum of degrees is odd**

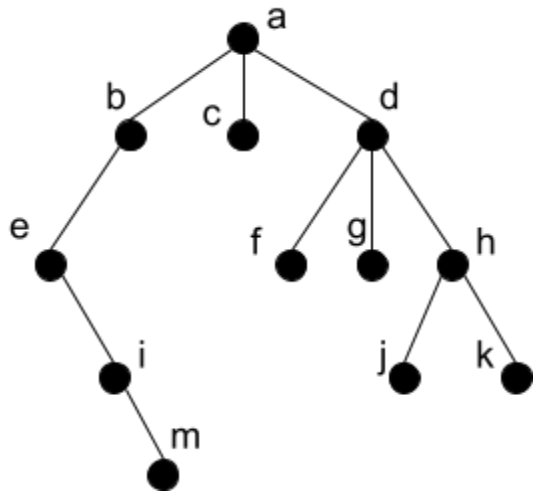
c) 6, 4, 4, 2, 2, 2, 2 **Yes. Graph on last page.**

Problem 5 [4 points]

For each **graphic** sequence in problem 4 (ignore any that were non-graphic), state whether or not they have an Euler circuit. If it does not, state why. If it does, write out a possible Euler circuit (label your vertices with letters a,b,c,...).
RUBRIC: 1 point for answer, 1 point for correct drawing/circuit or justification
5a) No. Not all vertices have even degree
6a) Ye. Answer on last page

Problem 6 [5 points]

Let T be the following rooted tree.



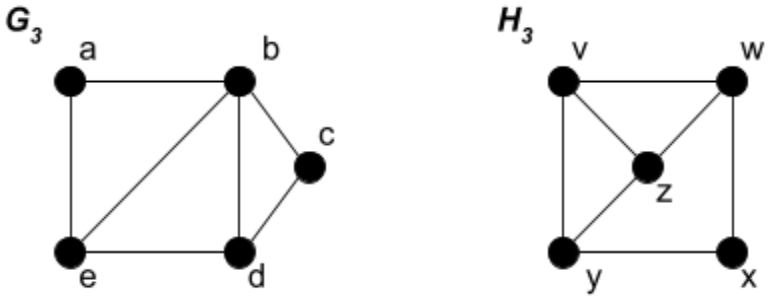
a) What is the height of T? **RUBRIC: 1 point**
4

b) What are the leaves of T?
RUBRIC: 2 points if correct, 1 if off by 1
{c,f,g,j,k,m}

c) What are the descendents of d?
RUBRIC: 2 points if correct, 1 if off by 1
{f,g,h,j,k}

Problem 7 (Extra Credit) [5 points]

Let G_3 and H_3 be the following graphs:

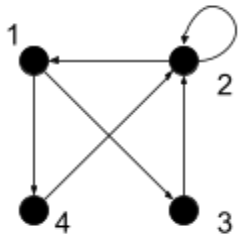


a) Both of the above graphs can be assigned a valid 3-coloring. List out the five vertices for each graph and assign each one a color from {red, blue, yellow} that results in a valid 3-coloring.
RUBRIC: 1.5 points per graph. Must be perfect.
a: A
b: B
c: A/C (must be different from d)
d: A/C (must be different from c)
e: B

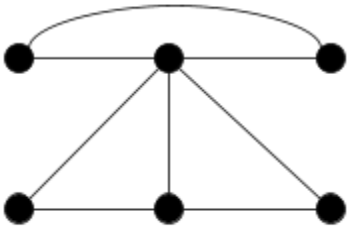
v: A
w: B
x: A/C
y: B
z: C

b) Are G_3 and H_3 isomorphic? If so, provide the isomorphism. If not, state why.
RUBRIC: 1 point for answer. 1 point for justification.
No. Vertices have different degrees.

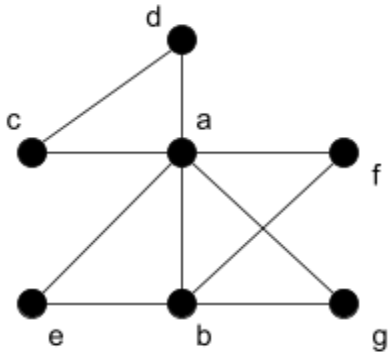
Answer to 2a) G_2



Graph for 4a)



Graph for 4c)



Potential answer to 5 (any path that crosses EVERY edge and starts and stops at the same place):

a,d,c,a,e,b,a,g,b,g,a