Part 1 - Propositional Logic [2 Points]

1. Let r and s be the following propositions: [2 points]

   r: I buy a movie ticket
   s: I have $20

   Write each of the following propositions using r, s, and logical operators

   a) Having $20 is a sufficient condition for me to buy a movie ticket

      [1 point] $s \rightarrow r$

   b) I do not have $20 but I still buy a movie ticket

      [1 point] $\neg s \land r$

Part 2 - Predicates and Quantifiers [4 Points]

2. Write the negation of the following proposition and simplify it to the point where all negation symbols directly precede a predicate: [2 points]

   $$\exists x \forall z \forall y, (P(x) \land Q(y,z))$$

   [2 points if perfect] $\forall x \exists z \exists y(\neg P(x) \lor \neg Q(x))$
   [1 point if not fully simplified] $\forall x \exists z \exists y \neg(P(x) \land Q(x))$

3. Consider the following predicates, both with domain $\mathbb{N}$: [2 points]

   P(x): x is an even number
   Q(x): I like the number x

   Write the logical expression for the statement: **I like all odd numbers**

   [2 points if perfect] $\forall x(\neg P(x) \rightarrow Q(x))$

Part 3 - Logical Equivalence [2 Points]

4. Select the logical expression that is equivalent to: **$\neg(p \lor q) \rightarrow (p \lor r)$** [2 points]

    a) $\neg(p \land q \land \neg r)$

    b) $p \lor (q \land r)$

    c) $p \land (q \lor r)$

    d) $p \lor q \lor r$

    e) $\neg p \lor (\neg q \land r)$

Part 4 - Logical Reasoning and Proofs [2 Points]

5. Find the assignment of truth values to **p**, **q** and **r** that prove that argument below is invalid. [2 points]
   [2 points if perfect]

$q \rightarrow p$
$r \rightarrow q$
r

$\therefore \neg q$

p	**T**
q	**T**
r	**T**

Part 5 - Sets [8 Points]

6. Select **ALL** expressions that evaluate to FALSE: [2 points]
   [2 points if perfect][1 point if off by one]
   a) **$\{ \varnothing \} \in \{ \varnothing \}$**

   b) $\{4\} \subseteq \{4\}$

   c) $6 \in \{3, 4, 5\} \cup \{4, 5, 6\}$

   d) $\{2, 4, 6\} \in \mathcal{P}(\{1, 2, 3, 4, 5, 6\})$

   e) **$\mathcal{P}(\{1, 2, 3, 4\}) \subseteq \mathcal{P}(\{1, 2, 3\})$**

7. Write the cardinality of each of the following sets: [2 points]

   a) $\{ x \mid x \in \mathbb{N}, x \leq 90 \text{ and } x \bmod 5 = 2 \}$
   [2 points if perfect]
   $|\{7, 12, 17, 22, \ldots, 87\}| = 17$

   b) $\{ \{x, y\}, x, \{x, \{y, z\}\}, z \}$
   [2 points if perfect]
   4

8. Let $S_i = \{ i, 2i, 4i \}$. Compute the following: [4 points]

   a) $\bigcup_{i=0}^{3} S_i$

   [2 points if perfect]
   $\{0, 1, 2, 3, 4, 6, 8, 12\}$

   b) $\bigcap_{i=1}^{2} S_i$
   [2 points if perfect]
   $\{2,4\}$

Part 6 - Relations and Functions [10 points]

9. The domain for the relation ***R*** is on the set of all integers, where **(*x*, *y*)** ∈ ***R*** if ***x*** divides ***y***. Which of the following are true? Circle all that apply. [2 points]
[2 points if perfect][1 point if off by one]
Note: Assume 0 divides 0.

   a) ***R* is reflexive**

   b) ***R*** is symmetric

   c) ***R* is antisymmetric**

   d) ***R* is transitive**

   e) None of the above

10. For the relation ***R*** in the previous question, state whether it is a partial ordering, a total ordering, an equivalence relation, or none of the above, and explain why. [2 points]
[2 points if perfect][1 point if answer is total ordering]

Partial ordering. It is reflexive, antisymmetric, transitive, but not complete

11. Let f: {1, 2, 3, 4, 5} → ℤ, be defined by f = {(1,1), (2,4), (3,9), (4,16), (5,25)}. What is the range of f? [2 points]
[2 points if perfect]
{1,4,9,16,25}

12. Let f: ℤ → ℤ by defined by f(x) = 3x - 12. Which of the following are true? Circle all that apply. [2 points]
[2 points if perfect][1 point if off by one]

   a) f is injective (one-to-one)

   b) f is surjective (onto)

   c) f is bijective

   d) f^{-1} is a function

13. Evaluate the following expressions [2 points]

    a) 13 mod 4 **[1 point] 1**

    b) 167 mod 5 **[1 point] 2**

Part 7 - Sequences [10 points]

14. Consider the following recurrence relation: [3 points]

$a_n = 4$ if $n \leq 3$

$a_n = a_{\lfloor n/4 \rfloor} + 10$

The initial index for { a_k } is 0

a) Compute the term at index 17 **[1 point] 24**

b) What is the largest base case index? **[1 point] 3**

c) What is the smallest base case index? **[1 point] 0**

15. Compute the following sum: [2 points]

$$\sum_{i=0}^{3} \sum_{j=3}^{5} (i - j)$$

[2 points if correct answer, or correct work with minor addition error]

(-3 + -4 + -5) + (-2 + -3 + -4) + (-1 + -2 + -3) + (0 + -1 + -2) = -30

16. Prove using mathematical induction that for all positive integers, n: [5 points]

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

Do not omit any required steps

[1 point]
Let P(n): $1^3 + 2^3 + \ldots + n^3 = n^2(n+1)^2 / 4$
Must prove P(n) is TRUE for all n ≥ 1

[1 point]
Base case: P(1): $1^3 = 1^2(1 + 1)^2 / 4 = 4 / 4 = 1$. TRUE

[1 point]
Assume P(k): $1^3 + 2^3 + \ldots + k^3 = k^2(k+1)^2 / 4$ is TRUE for arbitrary k ≥ 2
Must show P(k+1): $1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = (k+1)^2(k+2)^2 / 4$ is TRUE

[1 point]
$1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = k^2(k+1)^2 / 4 + (k+1)^3$ by inductive assumption

$= (k^2(k+1)^2 + 4(k+1)^3) / 4 = (k^2(k+1)^2 + (k+1)^2\, 4(k+1))/4 = (k+1)^2(k^2 + 4k + 4)/4$

$= (k+1)^2(k+2)^2/4$

[1 point]
Therefore P(n) is TRUE for all n ≥ 1

Part 8 - Counting Methods [6 points]

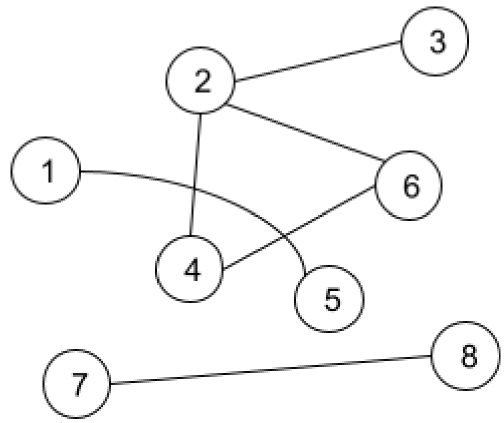
For each of the following questions you must explain your answer to receive full credit.

17. You and your friends are going to carpool to the Adirondacks for a summer hiking trip. There are 10 of you total, and 2 cars, each of which can hold 5 people. Two of your friends, Alice and Bob, are newly dating and refuse to be put in separate cars. How many ways can you split the group into the two cars for your trip? [3 points]
    [1 point for correct answer]
    8 * 7 * 6 / 3! = 56 (or C(8,3))
    [2 points for reasonable explanation]
    (Alice and Bob go together, then there are 8 ways to pick the third person in their car, 7 to pick the 4th, and 6 to pick the 5th. But order doesn't matter so divide by 3!)

18. Three candidates are interviewing for a job. The eight interviewers rank the three candidates in order of preference. Without having seen the interviewer feedback, can you guarantee that there are at least two interviewers that completely agree in their ranking? Why or why not? [3 points]
    [1 point for correct answer]
    Yes.
    [2 points for reasonable explanation]
    There are 3 * 2 * 1 = 6 ways to rank the candidates, and 8 people ranking them. So by pigeonhole, at least two interviews must have the same ranking

Part 9 - Graphs [6 points]

19. Consider the following graph: [3 points]



a) What vertices are in the same connected component as 6?
[1 point] 4, 2, 3

b) Which vertex has the largest degree?
[1 point] 2

c) Does this graph contain a cycle?
[1 point] Yes

20. Consider the following degree sequence for graph ***G***: 4, 4, 3, 3, 1, 1 [3 points]

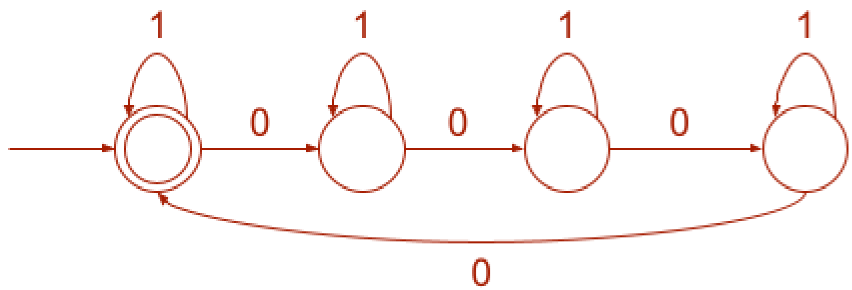
a) How many edges does ***G*** have?
[1 point] 8

b) If ***G*** is connected, is it guaranteed to have an Euler circuit? Why or why not?
[1 point] No
[1 point] All degrees are not even

Part 10 - Finite Automata (Extra Credit) [5 points]

21. Construct a DFA that accepts a binary string iff the number of 0's in the string is a multiple of 4. For example, it must accept "0000", "010001", "1111", and "01101111010". [5 points]

    [5 points if perfect]



Scrap Paper