CSE 191

Introduction to Discrete Structures

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Counting Methods

Outline

- **Product and Sum Rule**
- Permutations
- Combinations
- Pigeonhole Principle

Product Rule

The Product Rule

Suppose that a procedure can be broken down into a sequence of 2 tasks.

If there are n_1 ways to do the first task, and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

The product rule can be phrased in terms of sets:

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then $|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$

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Let $|\boldsymbol{A_1}| = \boldsymbol{k_1}$, then we have $\boldsymbol{k_1}$ ways to pick the first item in the tuple…

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Let $|A_1| = k_1$, then we have k_1 ways to pick the first item in the tuple…

Let $|A_2| = k_2$, then we have k_2 ways to pick the second item in the tuple…

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Let $|\boldsymbol{A_2}| = \boldsymbol{k_2}$, then we have $\boldsymbol{k_2}$ ways to pick the second item in the tuple…

…

Let $|\boldsymbol{A_n}| = \boldsymbol{k_n}$, then we have $\boldsymbol{k_n}$ ways to pick the first item in the tuple…

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Let $|A_1| = k_1$, then we have k_1 ways to pick the first item in the tuple…

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…

Let $|A_n| = k_n$, then we have k_n ways to pick the first item in the tuple…

Total Number of Possible Tuple: $k_1 \cdot k_2 \cdot \ldots \cdot k_n$

Product Rule Example

Consider the process of making a cake, in which we choose a flavor, a filling, a frosting, and a lettering color from the following options:

Flavors = {vanilla, chocolate, swirl}
Fillings = {raspberry, custard, whipped cream, ganache}
Frostings = {whipped cream, butter cream}
Colors = {red, blue, orange, green, purple}

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Set of all possible cakes: Flavors × Fillings × Frostings × Colors

Number of possible cakes: |Flavors × Fillings × Frostings × Colors| = |Flavors| · |Fillings| · |Frostings| · |Colors| = 3 · 4 · 2 · 5 = 120 different cakes!

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How many different sequences of 0s and 1s (bit strings) can we make of length n?

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How many choices do we have for the first character? **2**

For the second character? Third? Fourth? nth?

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For the second character? Third? Fourth? nth? **2**

$2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ possible bit strings of length ***n***.

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How many rows did we have in our truth tables with *n* variables…?

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$2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ possible bit strings of length ***n***.

How many rows did we have in our truth tables with *n* variables…? 2^n

…each row was a unique combination of T/F

Product Rule Example

Consider a small company with 5 employees: 2 managers, 3 workers.

We want to plan a lunch break as follows:

- Only one employee can be on break at a time
- The first and last person will be a manager
- The middle 3 breaks will be the 3 works

How many ways can we create a lunch schedule?

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2 * 3 * 2 * 1 * 1 = 12 different schedules

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How many ways can we create a lunch schedule?

2 * 3 * **2** * 1 * 1 = 12 different schedules

Number of ways we can pick the second worker

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Consider a small company with 5 employees: 2 managers, 3 workers.

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Number of ways we can pick the third worker

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How many ways can we create a lunch schedule?

2 * 3 * 2 * 1 * 1 = 12 different schedules

Number of ways we can pick the second manager

Product Rule Examples

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How many different license plates can be made if each plate is a sequence of 3 uppercase English letters followed by 3 digits (0-9)?

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How many different license plates can be made if each plate is a sequence of 3 uppercase English letters followed by 3 digits (0-9)?

26 * 26 * 26 * 10 * 10 * 10 = 17,576,000

Sum Rule

The Sum Rule

If a task can be done either in one of n_1 ways **OR** in one of n_2 ways, where none of the of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

The sum rule can be phrased in terms of sets:

Let $A_1, A_2, \ldots, A_n$ be **mutually disjoint**. Then $|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1|+|A_2|+\ldots+|A_n|$

Sum Rule

Note that the sum rule does not account for overlap. The sets of ways to complete the task must be mutually disjoint.

If we want to handle overlap, we can use the **Inclusion-Exclusion Principle**

Subtraction Rule

Inclusion-Exclusion Principle (or the Subtraction Rule)

If a task can be done in n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways that are common to to the two different ways.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Sum Rule Example

The sum rule is ***usually*** used in conjunction with the product rule.

Consider furnishing your living room with a TV, Seating, and a Table

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There are three types of TVs in your budget

- LED (42in, 55in, 60in, 65in)
- OLED (24in, 30in)
- 4K LED (50in, 55in)

How many ways can we pick the ONE TV we will buy?

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How many ways can we pick the ONE TV we will buy? 4 + 2 + 2 = 8

Sum Rule Example

The sum rule is ***usually*** used in conjunction with the product rule.

Consider furnishing your living room with a TV, Seating, and a Table

There are two types of seating in you are considering

- Recliner (rocking, stationary)
- Couch (3 seater, 2 seater)

How many ways can you pick the ONE type of seating you want?

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There are two types of seating in you are considering
- Recliner (rocking, stationary)
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How many ways can you pick the ONE type of seating you want? 2 + 2 = 4

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There are three options for an end table: square, round, hexagonal

Sum Rule Example

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Consider furnishing your living room with a TV, Seating, and a Table

You have 8 ways to pick a TV, 4 to pick seating, and 3 to pick a table.

How many ways can you furnish the room?

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You have 8 ways to pick a TV, 4 to pick seating, and 3 to pick a table.

How many ways can you furnish the room?

|TVs| · |Seating| · |Tables| **(product rule)**

(4 + 2 + 2) · (2 + 2) · 3 = 96 **(apply sum rule on individual sets)**

Sum Rule Example

Imagine a programming language where variable names can be one or two characters long. The first character must be a lowercase letter, and the second character (if it exists) must be a lowercase letter or one of the 10 digits. There are 5 reserved keywords in the programming language. How many possible variable names exist?

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Number of 1 character long variable names?

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Number of 1 character long variable names? **26**

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Number of 2 character long variable names?

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Number of 2 character long variable names? **26 * (26 + 10)**

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Number of 1 character long variable names? **26**

Number of 2 character long variable names? **26 * (26 + 10)**

Number of 1 OR 2 character long names?

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Number of 1 character long variable names? **26**

Number of 2 character long variable names? **26 * (26 + 10)**

Number of 1 OR 2 character long names? **26 + 26 * 36**

Sum Rule Example

Imagine a programming language where variable names can be one or two characters long. The first character must be a lowercase letter, and the second character (if it exists) must be a lowercase letter or one of the 10 digits. There are 5 reserved keywords in the programming language. How many possible variable names exist?

Number of 1 character long variable names? **26**

Remove the reserved keywords

Number of 2 character long variable names? **26 * (26 + 10)**

Number of 1 OR 2 character long names? **26 + 26 * 36 - 5**

Outline

- Product and Sum Rule
- **Permutations**
- Combinations
- Pigeonhole Principle

Permutations

A **permutation** of a set of *distinct* objects is an **ordered arrangement** of these objects.

An ordered arrangement of ***r*** elements of a set is called an ***r*-permutation**.
The number of ***r*-permutations** of a set with ***n*** elements is denoted by ***P*(*n*,*r*)** or ***nPr***.

Permutation Example

How many ways can we assign distinct roles to 3 students from a group of 5 to help out with a demo?

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How many ways can we assign distinct roles to 3 students from a group of 5 to help out with a demo?

Select any of the 5 students for the first role.

Then choose any of the 4 remaining students for the second role.

Finally choose one of the remaining 3 students for the last role.

Total arrangements: $5 * 4 * 3 = 60$

Permutation Example

How many ways can we assign distinct roles to 3 students from a class of 100 to help out with a demo?

Select any of the 100 students for the first role.

Then choose any of the 99 remaining students for the second role.

Finally choose one of the remaining 98 students for the last role.

Total arrangements: 100 * 99 * 98 = 970,200

We've just computed P(100, 3) (or 100 P 3), the number of 3-permutations for a set of 100 elements.

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Let n be any positive integer. How many ways can n people form a line?

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$$n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot 2 \cdot 1$$

Permutations

Let ***n*** be any positive integer. How many ways can ***n*** people form a line?

$$n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot 2 \cdot 1 = n!$$

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ n \cdot (n-1)! & \text{if } n > 1 \end{cases}$$

Factorial

Let $\boldsymbol{n} \geq 0$ be an integer. The **factorial** of $\boldsymbol{n}$, denoted by $\boldsymbol{n!}$ is defined by:

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot 2 \cdot 1$$

Note: For convenience, we define **0! = 1.**

We can also write it as a recurrence relation:

$$a_0 = 1$$

$$a_n = n \cdot a_{n-1} \text{ for } n > 0$$

Permutations and Factorial

Theorem

If n is a positive integer, and r is an integer s.t. $1 \leq r \leq n$, then there are

$$P(n,r) = n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot (n - r + 1)$$

***r*-permutations** of a set with n distinct elements

Permutations and Factorial

The number of ***r*-permutations** from a set with ***n*** elements:

$$P(n, r) = \frac{n!}{(n-r)!}$$

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Remember: **1 ≤ *r* ≤ *n***

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$$= n(n-1)\cdots(n-r+1)$$

Permutations and Factorial

The number of **permutations** from a set with ***n*** elements:

$$P(n, n) = \frac{n!}{(n-n)!} = n!$$

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John is planning a party. He must first shop for food and decorations at 3 different stores, but he can go to the stores in any order he wants. He must then decorate the 5 rooms of his house in any order. How many ways can John prepare for the party?

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$\boldsymbol{P(3,3) \cdot P(5,5) = 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720}$

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Note the use of product rule!

$$P(3,3) \cdot P(5,5) = 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 = 720$$

Outline

- Product and Sum Rule
- Permutations
- **Combinations**
- Pigeonhole Principle

Combinations

A **combination** of a set of *distinct* objects is an **unordered arrangement** of these objects. An ***r*-combination** is simply a subset with ***r*** elements

The number of ***r*-combinations** of a set with ***n*** elements is denoted by

C*(*n*,*r*)** or ***nCr or $\binom{n}{r}$. Sometimes referred to as ***n*** choose ***r***.

Combinations

How many ways can I select 3 students from a group of 5 to help with a demo? This time we don't have specific roles for the students.

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Previously, we had specific roles, so the arrangement of students was **ordered** (it was a permutation). So Alice, Bob, Carl was a different permutation than Bob, Carl, Alice.

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Now we are making an **unordered** selection. So Alice, Bob, and Carl is the same selection (same subset) as Bob, Carl, Alice.

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How can we go from number of permutations to number of combinations?

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Combinations

How many ways can I select 3 students from a group of 5 to help with a demo? This time we don't have specific roles for the students.

Consider selecting Alice, Bob, and Carl as the three students. When we were counting permutations, we considered every possible ordering of Alice, Bob, and Carl as a distinct permutation. How many orderings of Alice Bob and Carl are there?

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Combinations

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Consider selecting Alice, Bob, and Carl as the three students. When we were counting permutations, we considered every possible ordering of Alice, Bob, and Carl as a distinct permutation. How many orderings of Alice Bob and Carl are there? $\boldsymbol{P(3,3) = 3 \cdot 2 \cdot 1 = 6}$

That means there are 6 permutations that contain Alice, Bob, and Carl, but in terms of combinations, we only want to count this group once.

Combinations

How many ways can I select 3 students from a group of 5 to help with a demo? This time we don't have specific roles for the students.

$P(5,3) = 60$. But each combination shows up 6 times.

$C(5,3) = P(5,3)/6 = 10$

Combinations

Theorem

For any non-negative integers ***n*** and ***r*** s.t. **0 ≤ *r* ≤ *n***:

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 2 \cdot 1}$$

Combinations

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$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 2 \cdot 1}$$

…Therefore **C(5,3) = 5!/(3!2!) = 120 / (6*2) = 10**

Examples

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Since we only care about strings that contain ABC together in that order, consider ABC a single character. Now there are 6 total characters.

$P(6,6) = 6! = 720$

Examples

Suppose that there are 9 math faculty and 11 computer science faculty. How many ways can we create a committee to develop a discrete math curriculum if we want the committee to have 3 faculty from each dept?

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Suppose that there are 9 math faculty and 11 computer science faculty. How many ways can we create a committee to develop a discrete math curriculum if we want the committee to have 3 faculty from each dept?

$C(9,3) * C(11,3) = ((9 * 8 * 7)/6) * ((11 * 10 * 9) / 6) = 13{,}860$

Example

The NFL has 32 teams, divided into 2 conferences of 16. 7 teams from each conference make the playoffs each season. If we assume the Bills make the playoffs, how many different possible sets of remaining playoff teams are there?

Example

The NFL has 32 teams, divided into 2 conferences of 16. 7 teams from each conference make the playoffs each season. If we assume the Bills make the playoffs, how many different possible sets of remaining playoff teams are there?

$C(15,6) * C(16,7) = 15! / (6!\ 9!) * 16! / (6!\ 9!) = 400{,}800{,}400$

Example

The NFL has 32 teams, divided into 2 conferences of 16. 7 teams from each conference make the playoffs each season. If we assume the Bills make the playoffs, how many different possible sets of remaining playoff teams are there?

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$$P(7,7) * P(7,7) = 7! * 7! = 25{,}401{,}600$$

Another Point of View

Let's say we want to know the number of permutations of the first 8 letters of the alphabet: A,B,C,D,E,F,G,H. There are 8! such permutations.

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There are 3 leftover characters…and 3! = 6 ways to arrange these three characters. So "A H B E D" shows up as the first 5 characters in 6 of our 8-permutations.

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Furthermore…there are 5! = 60 ways to arrange these 5 characters in different orders. If we do not care about the order that they occur in, then we can divide by 60 to treat all permutations where A H B E D appear as the first 5 characters in any order as one combination.

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Therefore, if we divide 8! by 3! and by 5! we get C(8,5)

Outline

- Product and Sum Rule
- Permutations
- Combinations
- **Pigeonhole Principle**

Pigeonhole Principle

The **<u>Pigeonhole Principle</u>** is a **very** simple idea, but can be applied in many clever ways to prove some surprising things…

https://commons.wikimedia.org/wiki/File:TooManyPigeons.jpg#/media/File:TooManyPigeons.jpg, https://en.wikipedia.org/wiki/Pigeonhole_principle

Pigeonhole Principle

Pigeonhole Principle: If you put k pigeons into n pigeonholes, with $k > n$, then at least one pigeonhole contains at least two pigeons.

In terms of functions: If f: $A \rightarrow B$ where the codomain has size $|B| = n$ and the domain $|A| = k$ where $k > n$, then f must map at least 2 domain items to the same codomain element.

Alternate Wording and Proof

An equivalent way to state the pigeonhole principle:

If n is a positive integer, and $n + 1$ or more elements are put into n boxes, then there is at least one box containing two or more objects.

Proof (by contradiction):

1. Assume no box contains more than one object
2. Then by our assumption the total number of objects is at most n
3. This is a contradiction since there are at least $n + 1$ objects

Pigeonhole Principle Example

Prove that if 7 distinct numbers are selected from ***S* = {1,2,...11}**, then two of those numbers must sum to 12.

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How can we phrase this in terms of the pigeonhole principle?

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Proof

- Consider the pairs of numbers in ***S*** that sum to 12:
  - (1,11), (2,10), (3,9), (4,8), (5,7), 6

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- The box with 2 values contain two numbers which sum to 12

Pigeonhole Principle

<u>**Generalized Pigeonhole Principle:**</u>

If you put k objects into n boxes, then at least one box contains at least $\lceil k/n \rceil$ objects.

Basically, you cannot put a fraction of an item in a box (or more gruesomely…you cannot split up one pigeon into multiple boxes).

The fractional item gets rounded up (ceiling function)

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200 total students: $\boldsymbol{k = 200}$

Placed in 60 groups (boxes): $\boldsymbol{n = 60}$

Largest group = $\lceil 200/60 \rceil = \lceil 3.333 \rceil = 4$ students in the largest group

Pigeonhole Principle Example

A round-robin tournament is a tournament where each player plays each of the other players exactly once.

Prove that if there are $n > 2$ total players, and everyone wins at least one game, then there must be two players with the same number of wins.

Pigeonhole Principle Example

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- Everyone wins at least one game
    - Possible win totals for each player: 1, 2, 3, …, (n-1)
    - So (n - 1) different win totals
- By the pigeonhole principle, there are ***n*** players, and ***n* - 1** possible number of wins, so at least 2 people must have the same win record

Contrapositive Pigeonhole Principle

Contrapositive of the generalized pigeonhole principle:

Suppose you have ***k*** elements and ***n*** boxes. In order to guarantee that there is a box that contains at least ***b*** items, ***k*** must be at least ***n**(b-1) + 1.**

Example

How many people do we need to have in a room to guarantee that at least 3 people in the room were born on the same day of the year?

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What are our elements (***k***)? The people in the room

What are our boxes (***n***)? The days of the year

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How many people do we need to have in a room to guarantee that at least 3 people in the room were born on the same day of the year?

***n* = 366** possible birth dates

Want at least ***b* = 3** people with the same birth date (in the same box)

Let $k = n * (b - 1) + 1 = 733$

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***n* = 366** possible birth dates

Want at least ***b* = 3** people with the same birth date (in the same box)

Let $\boldsymbol{k = n * (b - 1) + 1 = 733}$

Note: $\lceil \mathbf{733/366} \rceil = \lceil \mathbf{2.002} \rceil = \mathbf{3}$.

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How many students must be in a class to guarantee that at least two students receive the same score on the final exam if it is graded 0-100?

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5 possible grades, $\boldsymbol{k = 5 * (6 - 1) + 1 = 26}$