Outline

- **Product and Sum Rule**
- Permutations
- Combinations
- Pigeonhole Principle
The Product Rule

Suppose that a procedure can be broken down into a sequence of 2 tasks.

If there are $n_1$ ways to do the first task, and for each of these ways of doing the first task, there are $n_2$ ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

The product rule can be phrased in terms of sets:

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then $|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$
Let $A_1, A_2, ..., A_n$ be finite sets. Then $|A_1 \times A_2 \times ... \times A_n| = |A_1| \cdot |A_2| \cdot ... \cdot |A_n|$
Let $A_1, A_2, \ldots, A_n$ be finite sets. Then $|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$

Let $|A_1| = k_1$, then we have $k_1$ ways to pick the first item in the tuple...
Product Rule

Let \( A_1, A_2, \ldots, A_n \) be finite sets. Then 
\[
|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|
\]
Let \( |A_1| = k_1 \), then we have \( k_1 \) ways to pick the first item in the tuple...

Let \( |A_2| = k_2 \), then we have \( k_2 \) ways to pick the second item in the tuple...
Product Rule

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then $|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$

Let $|A_1| = k_1$, then we have $k_1$ ways to pick the first item in the tuple...

Let $|A_2| = k_2$, then we have $k_2$ ways to pick the second item in the tuple...

... 

Let $|A_n| = k_n$, then we have $k_n$ ways to pick the first item in the tuple...
Product Rule

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Let $|A_1| = k_1$, then we have $k_1$ ways to pick the first item in the tuple...

Let $|A_2| = k_2$, then we have $k_2$ ways to pick the second item in the tuple...

... 

Let $|A_n| = k_n$, then we have $k_n$ ways to pick the first item in the tuple...

Total Number of Possible Tuple: $k_1 \cdot k_2 \cdot \cdots \cdot k_n$
Product Rule Example

Consider the process of making a cake, in which we choose a flavor, a filling, a frosting, and a lettering color from the following options:

- Flavors = \{vanilla, chocolate, swirl\}
- Fillings = \{raspberry, custard, whipped cream, ganache\}
- Frostings = \{whipped cream, butter cream\}
- Colors = \{red, blue, orange, green, purple\}
Consider the process of making a cake, in which we choose a flavor, a filling, a frosting, and a lettering color from the following options:

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Each possible cake is a tuple, ie: (swirl, custard, butter cream, red)
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Set of all possible cakes: Flavors $\times$ Fillings $\times$ Frostings $\times$ Colors
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Each possible cake is a tuple, ie: (swirl, custard, butter cream, red)

Set of all possible cakes: Flavors \times Fillings \times Frostings \times Colors

Number of possible cakes: |Flavors \times Fillings \times Frostings \times Colors| = |Flavors| \cdot |Fillings| \cdot |Frostings| \cdot |Colors| = 3 \cdot 4 \cdot 2 \cdot 5 = 120 different cakes!
Product Rule Example

How many different sequences of 0s and 1s (bit strings) can we make of length $n$?
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How many different sequences of 0s and 1s (bit strings) can we make of length $n$?

How many choices do we have for the first character?
How many different sequences of 0s and 1s (bit strings) can we make of length $n$?

How many choices do we have for the first character? 2

For the second character? Third? Fourth? nth?
How many different sequences of 0s and 1s (bit strings) can we make of length \( n \)?

How many choices do we have for the first character? \( 2 \)

For the second character? Third? Fourth? nth? \( 2 \)

\[ 2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n \] possible bit strings of length \( n \).
How many different sequences of 0s and 1s (bit strings) can we make of length $n$?

How many choices do we have for the first character? 2

For the second character? Third? Fourth? nth? 2

$2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ possible bit strings of length $n$.

How many rows did we have in our truth tables with $n$ variables...?
Product Rule Example

How many different sequences of 0s and 1s (bit strings) can we make of length $n$?

How many choices do we have for the first character? $2$
For the second character? Third? Fourth? nth? $2$

$2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ possible bit strings of length $n$.

How many rows did we have in our truth tables with $n$ variables...? $2^n$

...each row was a unique combination of T/F
Product Rule Example

Consider a small company with 5 employees: 2 managers, 3 workers.

We want to plan a lunch break as follows:

- Only one employee can be on break at a time
- The first and last person will be a manager
- The middle 3 breaks will be the 3 workers

How many ways can we create a lunch schedule?
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How many ways can we create a lunch schedule?

\[ 2 \times 3 \times 2 \times 1 \times 1 = 12 \text{ different schedules} \]
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How many ways can we create a lunch schedule?

\[2 * 3 * 2 * 1 * 1 = 12 \text{ different schedules}\]

**Number of ways we can pick the first manager**
Consider a small company with 5 employees: 2 managers, 3 workers.

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How many ways can we create a lunch schedule?

\[2 \times 3 \times 2 \times 1 \times 1 = 12\] different schedules

**Number of ways we can pick the first worker**
Consider a small company with 5 employees: 2 managers, 3 workers.

We want to plan a lunch break as follows:
- Only one employee can be on break at a time
- The first and last person will be a manager
- The middle 3 breaks will be the 3 works

How many ways can we create a lunch schedule?

\[ 2 \times 3 \times 2 \times 1 \times 1 = 12 \text{ different schedules} \]

Number of ways we can pick the second worker
Consider a small company with 5 employees: 2 managers, 3 workers.

We want to plan a lunch break as follows:

- Only one employee can be on break at a time
- The first and last person will be a manager
- The middle 3 breaks will be the 3 workers

How many ways can we create a lunch schedule?

$$2 \times 3 \times 2 \times 1 \times 1 = 12$$ different schedules

Number of ways we can pick the third worker
Consider a small company with 5 employees: 2 managers, 3 workers.

We want to plan a lunch break as follows:
- Only one employee can be on break at a time
- The first and last person will be a manager
- The middle 3 breaks will be the 3 works

How many ways can we create a lunch schedule?

\[2 \times 3 \times 2 \times 1 \times 1 = 12\] different schedules

**Number of ways we can pick the second manager**
The chairs in a lecture hall are labeled with an uppercase English letter followed by a positive 2 digit number. What is the maximum number of chairs we could label in this fashion?
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How many different license plates can be made if each plate is a sequence of 3 uppercase English letters followed by 3 digits (0-9)?
Product Rule Examples

The chairs in a lecture hall are labeled with an uppercase English letter followed by a positive 2 digit number. What is the maximum number of chairs we could label in this fashion? \(26 \times 99 = 2,574\)

How many different license plates can be made if each plate is a sequence of 3 uppercase English letters followed by 3 digits (0-9)?

\(26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000\)
The Sum Rule

If a task can be done either in one of \( n_1 \) ways \textbf{OR} in one of \( n_2 \) ways, where none of the of \( n_1 \) ways is the same as any of the \( n_2 \) ways, then there are \( n_1 + n_2 \) ways to do the task.

The sum rule can be phrased in terms of sets:

Let \( A_1, A_2, \ldots, A_n \) be \textbf{mutually disjoint}. Then \(|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|\)
Note that the sum rule does not account for overlap. The sets of ways to complete the task must be mutually disjoint.

If we want to handle overlap, we can use the **Inclusion-Exclusion Principle**.
Inclusion-Exclusion Principle (or the Subtraction Rule)

If a task can be done in \( n_1 \) ways or \( n_2 \) ways, then the number of ways to do the task is \( n_1 + n_2 \) minus the number of ways that are common to the two different ways.

\[
|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|
\]
The sum rule is *usually* used in conjunction with the product rule.

Consider furnishing your living room with a TV, Seating, and a Table.
Sum Rule Example

The sum rule is **usually** used in conjunction with the product rule.

Consider furnishing your living room with a TV, Seating, and a Table.

There are three types of TVs in your budget:
- LED (42in, 55in, 60in, 65in)
- OLED (24in, 30in)
- 4K LED (50in, 55in)

How many ways can we pick the ONE TV we will buy?
Sum Rule Example

The sum rule is *usually* used in conjunction with the product rule.

Consider furnishing your living room with a TV, Seating, and a Table.

There are three types of TVs in your budget:
- LED (42in, 55in, 60in, 65in)
- OLED (24in, 30in)
- 4K LED (50in, 55in)

How many ways can we pick the ONE TV we will buy? $4 + 2 + 2 = 8$
Sum Rule Example

The sum rule is *usually* used in conjunction with the product rule.

Consider furnishing your living room with a TV, Seating, and a Table.

There are two types of seating in you are considering:
- Recliner (rocking, stationary)
- Couch (3 seater, 2 seater)

How many ways can you pick the ONE type of seating you want?
The sum rule is usually used in conjunction with the product rule.

Consider furnishing your living room with a TV, Seating, and a Table.

There are two types of seating in you are considering:
- Recliner (rocking, stationary)
- Couch (3 seater, 2 seater)

How many ways can you pick the ONE type of seating you want? $2 + 2 = 4$
The sum rule is *usually* used in conjunction with the product rule.

Consider furnishing your living room with a TV, Seating, and a Table.

There are three options for an end table: square, round, hexagonal.
The sum rule is *usually* used in conjunction with the product rule.

Consider furnishing your living room with a TV, Seating, and a Table.

You have 8 ways to pick a TV, 4 to pick seating, and 3 to pick a table.

How many ways can you furnish the room?
The sum rule is *usually* used in conjunction with the product rule.

Consider furnishing your living room with a TV, Seating, and a Table.

You have 8 ways to pick a TV, 4 to pick seating, and 3 to pick a table.

How many ways can you furnish the room?

\[
|TVs| \cdot |Seating| \cdot |Tables| \quad \text{(product rule)}
\]

\[
(4 + 2 + 2) \cdot (2 + 2) \cdot 3 = 96 \quad \text{(apply sum rule on individual sets)}
\]
Imagine a programming language where variable names can be one or two characters long. The first character must be a lowercase letter, and the second character (if it exists) must be a lowercase letter or one of the 10 digits. There are 5 reserved keywords in the programming language. How many possible variable names exist?
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Number of 1 character long variable names?
Sum Rule Example

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Number of 1 character long variable names? 26
Imagine a programming language where variable names can be one or two characters long. The first character must be a lowercase letter, and the second character (if it exists) must be a lowercase letter or one of the 10 digits. There are 5 reserved keywords in the programming language. How many possible variable names exist?

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Number of 2 character long variable names?
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Number of 1 character long variable names? \(26\)

Number of 2 character long variable names? \(26 \times (26 + 10)\)
Imagine a programming language where variable names can be one or two characters long. The first character must be a lowercase letter, and the second character (if it exists) must be a lowercase letter or one of the 10 digits. There are 5 reserved keywords in the programming language. How many possible variable names exist?

Number of 1 character long variable names? 26

Number of 2 character long variable names? 26 * (26 + 10)

Number of 1 OR 2 character long names?
Imagine a programming language where variable names can be one or two characters long. The first character must be a lowercase letter, and the second character (if it exists) must be a lowercase letter or one of the 10 digits. There are 5 reserved keywords in the programming language. How many possible variable names exist?

Number of 1 character long variable names? 26

Number of 2 character long variable names? 26 * (26 + 10)

Number of 1 OR 2 character long names? 26 + 26 * 36
Imagine a programming language where variable names can be one or two characters long. The first character must be a lowercase letter, and the second character (if it exists) must be a lowercase letter or one of the 10 digits. There are 5 reserved keywords in the programming language. How many possible variable names exist?

Number of 1 character long variable names? 26

Number of 2 character long variable names? 26 \times (26 + 10)

Number of 1 OR 2 character long names? 26 + 26 \times 36 - 5
Outlines

- Product and Sum Rule
- **Permutations**
- Combinations
- Pigeonhole Principle
Permutations

A permutation of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of $r$ elements of a set is called an $r$-permutation. The number of $r$-permutations of a set with $n$ elements is denoted by $P(n,r)$ or $nPr$. 
Permutation Example

How many ways can we assign distinct roles to 3 students from a group of 5 to help out with a demo?
Permutation Example

How many ways can we assign distinct roles to 3 students from a group of 5 to help out with a demo?

Select any of the 5 students for the first role.

Then choose any of the 4 remaining students for the second role.

Finally choose one of the remaining 3 students for the last role.

Total arrangements: $5 \times 4 \times 3 = 60$
Permutation Example

How many ways can we assign distinct roles to 3 students from a class of 100 to help out with a demo?

Select any of the 100 students for the first role.

Then choose any of the 99 remaining students for the second role.

Finally choose one of the remaining 98 students for the last role.

Total arrangements: $100 \times 99 \times 98 = 970,200$

We've just computed $P(100, 3)$ (or $100 \text{ P } 3$), the number of 3-permutations for a set of 100 elements.
Permutations

Let $n$ be any positive integer. How many ways can $n$ people form a line?
Let $n$ be any positive integer. How many ways can $n$ people form a line?

\[ n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot 2 \cdot 1 \]
Let $n$ be any positive integer. How many ways can $n$ people form a line?

\[ n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot 2 \cdot 1 = n! \]

\[ n! = \begin{cases} 
1 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
(n \cdot (n - 1)!) & \text{if } n > 1 
\end{cases} \]
Factorial

Let $n \geq 0$ be an integer. The \textbf{factorial} of $n$, denoted by $n!$ is defined by:

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot 2 \cdot 1$$

\textbf{Note:} For convenience, we define $0! = 1$.

We can also write it as a recurrence relation:

$$a_0 = 1$$

$$a_n = n \cdot a_{n-1} \text{ for } n > 0$$
Theorem

If $n$ is a positive integer, and $r$ is an integer s.t. $1 \leq r \leq n$, then there are $r$-permutations of a set with $n$ distinct elements

$$P(n,r) = n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot (n - r + 1)$$
The number of \( r \)-permutations from a set with \( n \) elements:

\[
P(n, r) = \frac{n!}{(n - r)!}
\]
Permutations and Factorial

The number of \( r \)-permutations from a set with \( n \) elements:

\[
P(n, r) = \frac{n!}{(n - r)!}
= \frac{n(n - 1) \cdots (n - r + 1)(n - r)(n - r - 1) \cdots (2)(1)}{(n - r)(n - r - 1) \cdots (2)(1)}
\]
Permutations and Factorial

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P(n, r) = \frac{n!}{(n - r)!}
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\[
= \frac{n(n - 1) \cdots (n - r + 1)(n-r)(n-r-1)\cdots(2)(1)}{(n-r)(n-r-1)\cdots(2)(1)}
\]

Remember: \( 1 \leq r \leq n \)
The number of \( r \)-permutations from a set with \( n \) elements:

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P(n, r) = \frac{n!}{(n - r)!}
\]

\[
= \frac{n(n - 1) \cdots (n - r + 1)(n - r)(n - r - 1) \cdots (2)(1)}{(n - r)(n - r - 1) \cdots (2)(1)}
\]

\[
= n(n - 1) \cdots (n - r + 1)
\]

Remember: \( 1 \leq r \leq n \)
Permutations and Factorial

The number of permutations from a set with $n$ elements:

$$P(n, n) = \frac{n!}{(n - n)!} = n!$$
How many ways are there to select a first, second, and third prize winner from the top 10 hackathon projects?
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\[ P(10,3) = 10 \cdot 9 \cdot 8 = 720 \]
Permutations

How many ways are there to select a first, second, and third prize winner from the top 10 hackathon projects?

\[ P(10,3) = 10 \cdot 9 \cdot 8 = 720 \]

John is planning a party. He must first shop for food and decorations at 3 different stores, but he can go to the stores in any order he wants. He must then decorate the 5 rooms of his house in any order. How many ways can John prepare for the party?
How many ways are there to select a first, second, and third prize winner from the top 10 hackathon projects?

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John is planning a party. He must first shop for food and decorations at 3 different stores, but he can go to the stores in any order he wants. He must then decorate the 5 rooms of his house in any order. How many ways can John prepare for the party?

\[ P(3,3) \cdot P(5,5) = 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 \]
Permutations

How many ways are there to select a first, second, and third prize winner from the top 10 hackathon projects?

\[ P(10,3) = 10 \cdot 9 \cdot 8 = 720 \]

John is planning a party. He must first shop for food and decorations at 3 different stores, but he can go to the stores in any order he wants. He must then decorate the 5 rooms of his house in any order. How many ways can John prepare for the party?

\[ P(3,3) \cdot P(5,5) = 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 = 720 \]

Note the use of product rule!
Outline

- Product and Sum Rule
- Permutations
- **Combinations**
- Pigeonhole Principle
A **combination** of a set of **distinct** objects is an **unordered arrangement** of these objects. An **r-combination** is simply a subset with **r** elements.

The number of **r-combinations** of a set with **n** elements is denoted by

\[ C(n,r) \text{ or } nCr \text{ or } \binom{n}{r} \]. Sometimes referred to as **n choose r**.
How many ways can I select 3 students from a group of 5 to help with a demo? This time we don't have specific roles for the students.
How many ways can I select 3 students from a group of 5 to help with a demo? This time we don't have specific roles for the students.

Previously, we had specific roles, so the arrangement of students was **ordered** (it was a permutation). So Alice, Bob, Carl was a different permutation than Bob, Carl, Alice.
Combinations

How many ways can I select 3 students from a group of 5 to help with a demo? This time we don't have specific roles for the students.

Previously, we had specific roles, so the arrangement of students was ordered (it was a permutation). So Alice, Bob, Carl was a different permutation than Bob, Carl, Alice.

Now we are making an unordered selection. So Alice, Bob, and Carl is the same selection (same subset) as Bob, Carl, Alice.
Combinations

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Now we are making an unordered selection. So Alice, Bob, and Carl is the same selection (same subset) as Bob, Carl, Alice.

How can we go from number of permutations to number of combinations?
Combinations

How many ways can I select 3 students from a group of 5 to help with a demo? This time we don't have specific roles for the students.

Consider selecting Alice, Bob, and Carl as the three students. When we were counting permutations, we considered every possible ordering of Alice, Bob, and Carl as a distinct permutation. How many orderings of Alice Bob and Carl are there?
How many ways can I select 3 students from a group of 5 to help with a demo? This time we don't have specific roles for the students.

Consider selecting Alice, Bob, and Carl as the three students. When we were counting permutations, we considered every possible ordering of Alice, Bob, and Carl as a distinct permutation. How many orderings of Alice Bob and Carl are there? \( P(3,3) = 3 \cdot 2 \cdot 1 = 6 \)
Combinations

How many ways can I select 3 students from a group of 5 to help with a demo? This time we don't have specific roles for the students.

Consider selecting Alice, Bob, and Carl as the three students. When we were counting permutations, we considered every possible ordering of Alice, Bob, and Carl as a distinct permutation. How many orderings of Alice Bob and Carl are there? \( P(3,3) = 3 \cdot 2 \cdot 1 = 6 \)

That means there are 6 permutations that contain Alice, Bob, and Carl, but in terms of combinations, we only want to count this group once.
Combinations

How many ways can I select 3 students from a group of 5 to help with a demo? This time we don't have specific roles for the students.

\[ P(5,3) = 60. \] But each combination shows up 6 times.

\[ C(5,3) = P(5,3)/6 = 10 \]
Combinations

Theorem
For any non-negative integers \( n \) and \( r \) s.t. \( 0 \leq r \leq n \):

\[
C(n, r) = \frac{n!}{r!(n-r)!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 2 \cdot 1}
\]
Combinations

**Theorem**

For any non-negative integers $n$ and $r$ s.t. $0 \leq r \leq n$:

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 2 \cdot 1}
\]

...Therefore $\binom{5}{3} = \frac{5!}{(3!2!)} = \frac{120}{6 \cdot 2} = 10$
Examples

How many permutations of the letters ABCDEFGH contain substring ABC?
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Since we only care about strings that contain ABC together in that order, consider ABC a single character. Now there are 6 total characters.
How many permutations of the letters ABCDEFGH contain substring ABC?

Since we only care about strings that contain ABC together in that order, consider ABC a single character. Now there are 6 total characters.

\[ P(6,6) = 6! = 720 \]
Examples

Suppose that there are 9 math faculty and 11 computer science faculty. How many ways can we create a committee to develop a discrete math curriculum if we want the committee to have 3 faculty from each dept?
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Suppose that there are 9 math faculty and 11 computer science faculty. How many ways can we create a committee to develop a discrete math curriculum if we want the committee to have 3 faculty from each dept?

\[ C(9,3) \times C(11,3) = \frac{(9 \times 8 \times 7)}{6} \times \frac{(11 \times 10 \times 9)}{6} = 13,860 \]
Example

The NFL has 32 teams, divided into 2 conferences of 16. 7 teams from each conference make the playoffs each season. If we assume the Bills make the playoffs, how many different possible sets of remaining playoff teams are there?
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The NFL has 32 teams, divided into 2 conferences of 16. 7 teams from each conference make the playoffs each season. If we assume the Bills make the playoffs, how many different possible sets of remaining playoff teams are there?

\[ C(15,6) \times C(16,7) = \frac{15!}{(6! \times 9!)} \times \frac{16!}{(6! \times 9!)} = 400,800,400 \]
Example

The NFL has 32 teams, divided into 2 conferences of 16. 7 teams from each conference make the playoffs each season. If we assume the Bills make the playoffs, how many different possible sets of remaining playoff teams are there?

\[C(15,6) \times C(16,7) = \frac{15!}{(6! \times 9!)} \times \frac{16!}{(6! \times 9!)} = 400,800,400\]

In each conference the playoff teams are ranked 1-7. How many different ways can the 14 playoff teams be ranked?
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In each conference the playoff teams are ranked 1-7. How many different ways can the 14 playoff teams be ranked?

\[ P(7,7) \times P(7,7) = 7! \times 7! = 25,401,600 \]
Let's say we want to know the number of permutations of the first 8 letters of the alphabet: A, B, C, D, E, F, G, H. There are $8!$ such permutations.

One such permutation is:

A H B E D F C G
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This is one of those 5-permutations

There are 3 leftover characters...and 3! = 6 ways to arrange these three characters. So "A H B E D" shows up as the first 5 characters in 6 of our 8-permutations.
Another Point of View

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Furthermore...there are $5! = 60$ ways to arrange these 5 characters in different orders. If we do not care about the order that they occur in, then we can divide by 60 to treat all permutations where A, H, B, E, D appear as the first 5 characters in any order as one combination.
Another Point of View

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Therefore, if we divide 8! by 3! and by 5! we get C(8,5)
Outline

- Product and Sum Rule
- Permutations
- Combinations
- Pigeonhole Principle
Pigeonhole Principle

The **Pigeonhole Principle** is a very simple idea, but can be applied in many clever ways to prove some surprising things...

Pigeonhole Principle: If you put $k$ pigeons into $n$ pigeonholes, with $k > n$, then at least one pigeonhole contains at least two pigeons.

In terms of functions: If $f: A \rightarrow B$ where the codomain has size $|B| = n$ and the domain $|A| = k$ where $k > n$, then $f$ must map at least 2 domain items to the same codomain element.
An equivalent way to state the pigeonhole principle: If \( n \) is a positive integer, and \( n + 1 \) or more elements are put into \( n \) boxes, then there is at least one box containing two or more objects.

Proof (by contradiction):
1. Assume no box contains more than one object
2. Then by our assumption the total number of objects is at most \( n \)
3. This is a contradiction since there are at least \( n + 1 \) objects
Pigeonhole Principle Example

Prove that if 7 distinct numbers are selected from $S = \{1, 2, \ldots, 11\}$, then two of those numbers must sum to 12.
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How can we phrase this in terms of the pigeonhole principle?
Pigeonhole Principle Example

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**Proof**
- Consider the pairs of numbers in $S$ that sum to 12:
  - $(1,11), (2,10), (3,9), (4,8), (5,7), 6$
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Proof

- Consider the pairs of numbers in $S$ that sum to 12:
  - $(1,11), (2,10), (3,9), (4,8), (5,7), 6$
- Label 6 boxes with each of the pairs (or the single 6)
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  - \((1,11), (2,10), (3,9), (4,8), (5,7), 6\)
- Label 6 boxes with each of the pairs (or the single 6)
- Now consider selecting 7 distinct numbers from \( S \)
  - When a number is selected place it in the matching box
  - The pigeonhole principle states that at least one box will have 2 numbers
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- The box with 2 values contain two numbers which sum to 12
**Generalized Pigeonhole Principle:**

If you put $k$ objects into $n$ boxes, then at least one box contains at least $\lceil k/n \rceil$ objects.

Basically, you cannot put a fraction of an item in a box (or more gruesomely...you cannot split up one pigeon into multiple boxes).

The fractional item gets rounded up (ceiling function)
Students are asked to form groups for a project. There are 200 students asked to evenly form 60 groups. What size is the largest group formed?
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200 total students: $k = 200$

Placed in 60 groups (boxes): $n = 60$

Largest group = ⌈200/60⌉ = ⌈3.333⌉ = 4 students in the largest group
Pigeonhole Principle Example

A round-robin tournament is a tournament where each player plays each of the other players exactly once.

Prove that if there are \( n > 2 \) total players, and everyone wins at least one game, then there must be two players with the same number of wins.
Pigeonhole Principle Example

Prove that if there are $n > 2$ total players, and everyone wins at least one game, then there must be two players with the same number of wins.

- There are $n$ players, so each player plays $n - 1$ games
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  - So (n - 1) different win totals
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- Everyone wins at least one game
  - Possible win totals for each player: 1, 2, 3, ..., (n-1)
  - So $(n - 1)$ different win totals
- By the pigeonhole principle, there are $n$ players, and $n - 1$ possible number of wins, so at least 2 people must have the same win record
Contrapositive of the generalized pigeonhole principle:

Suppose you have $k$ elements and $n$ boxes. In order to guarantee that there is a box that contains at least $b$ items, $k$ must be at least $n*(b-1) + 1$. 
Example

How many people do we need to have in a room to guarantee that at least 3 people in the room were born on the same day of the year?
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What are our elements \( (k) \)? The people in the room

What are our boxes \( (n) \)? The days of the year
Example

How many people do we need to have in a room to guarantee that at least 3 people in the room were born on the same day of the year?

\[ n = 366 \text{ possible birth dates} \]

Want at least \( b = 3 \) people with the same birth date (in the same box)

Let \( k = n \times (b - 1) + 1 = 733 \)
Example

How many people do we need to have in a room to guarantee that at least 3 people in the room were born on the same day of the year?

\[ n = 366 \] possible birth dates

Want at least \( b = 3 \) people with the same birth date (in the same box)

Let \( k = n \times (b - 1) + 1 = 733 \)

Note: \( \lceil 733/366 \rceil = \lceil 2.002 \rceil = 3 \).
How many students must be in a class to guarantee that at least two students receive the same score on the final exam if it is graded 0-100?
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101 different possible scores, therefore we need at least 102 students to guarantee there's a tie.
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How many students must be in a class to guarantee that at least 6 will receive the same grade, if possible grades are A, B, C, D, and F?
Examples

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101 different possible scores, therefore we need at least 102 students to guarantee there's a tie.

How many students must be in a class to guarantee that at least 6 will receive the same grade, if possible grades are A, B, C, D, and F?

5 possible grades, \[ k = 5 \times (6 - 1) + 1 = 26 \]