CSE 191

Introduction to Discrete Structures

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Logical Equivalence

Outline

Logical Equivalence

- **Tautologies and Contradictions**
- Logical Equivalence
- Equivalence Laws
- Proving Logical Equivalence

Tautologies and Contradictions

A **tautology** is a compound proposition that is ALWAYS TRUE, no matter what the truth values of the propositional variables that occur in it.

A **contradiction** is a compound proposition that is ALWAYS FALSE.

A **contingency** is a compound proposition that is neither a contradiction or a tautology. There at least one assignment of truth values to the atomics that can result in TRUE, and at least one that can result in FALSE.

Tautologies and Contradictions Examples

p	$\neg p$	$p \lor \neg p$	$p \land \neg p$
F	T		
T	F		

Tautologies and Contradictions Examples

p	$\neg p$	$p \lor \neg p$	$p \land \neg p$
F	T	T	
T	F	T	

$p \lor \neg p$ is a tautology

Tautologies and Contradictions Examples

p	$\neg p$	$p \lor \neg p$	$p \land \neg p$
F	T	T	F
T	F	T	F

$p \lor \neg p$ is a tautology

$p \land \neg p$ is a contradiction

Tautologies and Contradictions Examples

Construct the truth table for the proposition f_1: $p \lor \neg(q \land p)$

Is f_1 a tautology, contradiction, or contingency?

p	q	$p \lor \neg(q \land p)$
F	F	
F	T	
T	F	
T	T	

Tautologies and Contradictions Examples

Construct the truth table for the proposition f_1: $p \lor \neg(q \land p)$

Is f_1 a tautology, contradiction, or contingency?

p	q	$(q \land p)$	$\neg(q \land p)$	$p \lor \neg(q \land p)$
F	F			
F	T			
T	F			
T	T			

Tautologies and Contradictions Examples

Construct the truth table for the proposition f_1: $p \lor \neg(q \land p)$

Is f_1 a tautology, contradiction, or contingency?

p	q	$(q \land p)$	$\neg(q \land p)$	$p \lor \neg(q \land p)$
F	F	F		
F	T	F		
T	F	F		
T	T	T		

Tautologies and Contradictions Examples

Construct the truth table for the proposition f_1: $p \lor \neg(q \land p)$

Is f_1 a tautology, contradiction, or contingency?

p	q	$(q \land p)$	$\neg(q \land p)$	$p \lor \neg(q \land p)$
F	F	F	T	
F	T	F	T	
T	F	F	T	
T	T	T	F	

Tautologies and Contradictions Examples

Construct the truth table for the proposition f_1: $p \lor \neg(q \land p)$

Is f_1 a tautology, contradiction, or contingency?

p	q	$(q \land p)$	$\neg(q \land p)$	$p \lor \neg(q \land p)$
F	F	F	T	T
F	T	F	T	T
T	F	F	T	T
T	T	T	F	T

Tautologies and Contradictions Examples

f_1 is therefore a tautology

We can also say that f_1 is equivalent to TRUE, $f_1 \equiv T$

p	q	$(q \wedge p)$	$\neg(q \wedge p)$	$p \vee \neg(q \wedge p)$
F	F	F	T	T
F	T	F	T	T
T	F	F	T	T
T	T	T	F	T

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- **Logical Equivalence**
- Equivalence Laws
- Proving Logical Equivalence

Logical Equivalence

Two propositions, ***p*** and ***q*** are **logically equivalent** if ***p*** ⇔ ***q*** is a tautology

Logical Equivalence

Two propositions, $\boldsymbol{p}$ and $\boldsymbol{q}$ are **logically equivalent** if $\boldsymbol{p} \Leftrightarrow \boldsymbol{q}$ is a tautology

- In other words, $\boldsymbol{p}$ and $\boldsymbol{q}$ are logically equivalent if their truth values in their truth table are all the same
- Two compound propositions are logically equivalent if their truth values agree for all combinations of the truth values of their atomics
- We write equivalence as $\boldsymbol{p} \equiv \boldsymbol{q}$
  - $\equiv$ is NOT a logical operator
  - $\boldsymbol{p} \equiv \boldsymbol{q}$ is NOT a compound proposition

Logical Equivalence Example

Are the propositions $p \rightarrow q$ and $q \lor \neg p$ logically equivalent?

p	q	$p \rightarrow q$	$\neg p$	$q \lor \neg p$	$(p \rightarrow q) \Leftrightarrow (q \lor \neg p)$
F	F				
F	T				
T	F				
T	T				

Logical Equivalence Example

Are the propositions $p \rightarrow q$ and $q \vee \neg p$ logically equivalent?

p	q	$p \rightarrow q$	$\neg p$	$q \vee \neg p$	$(p \rightarrow q) \Leftrightarrow (q \vee \neg p)$
F	F	T			
F	T	T			
T	F	F			
T	T	T			

Logical Equivalence Example

Are the propositions $p \rightarrow q$ and $q \lor \neg p$ logically equivalent?

p	q	$p \rightarrow q$	$\neg p$	$q \lor \neg p$	$(p \rightarrow q) \Leftrightarrow (q \lor \neg p)$
F	F	T	T		
F	T	T	T		
T	F	F	F		
T	T	T	F		

Logical Equivalence Example

Are the propositions $p \rightarrow q$ and $q \vee \neg p$ logically equivalent?

p	q	$p \rightarrow q$	$\neg p$	$q \vee \neg p$	$(p \rightarrow q) \Leftrightarrow (q \vee \neg p)$
F	F	T	T	T	
F	T	T	T	T	
T	F	F	F	F	
T	T	T	F	T	

Logical Equivalence Example

Are the propositions $p \rightarrow q$ and $q \vee \neg p$ logically equivalent?

p	q	$p \rightarrow q$	$\neg p$	$q \vee \neg p$	$(p \rightarrow q) \Leftrightarrow (q \vee \neg p)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

Logical Equivalence Example

p	q	$p \to q$	$\neg p$	$q \vee \neg p$	$(p \to q) \Leftrightarrow (q \vee \neg p)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

Columns 3 and 5 are identical, therefore the bidirectional implication of the two is always TRUE. Therefore $p \to q \equiv q \vee \neg p$.

Logical Equivalence Example

Are the propositions $p \oplus q$ and f_2: $(p \vee q) \wedge (p \rightarrow q)$ logically equivalent?

p	q	$p \oplus q$	$p \vee q$	$p \rightarrow q$	f_2	$(p \oplus q) \Leftrightarrow f_2$
F	F					
F	T					
T	F					
T	T					

Logical Equivalence Example

Are the propositions $p \oplus q$ and f_2: $(p \lor q) \land (p \rightarrow q)$ logically equivalent?

p	q	$p \oplus q$	$p \lor q$	$p \rightarrow q$	f_2	$(p \oplus q) \Leftrightarrow f_2$
F	F	F	F	T		
F	T	T	T	T		
T	F	T	T	F		
T	T	F	T	T		

Logical Equivalence Example

Are the propositions $p \oplus q$ and f_2: $(p \vee q) \wedge (p \rightarrow q)$ logically equivalent?

p	q	$p \oplus q$	$p \vee q$	$p \rightarrow q$	f_2	$(p \oplus q) \Leftrightarrow f_2$
F	F	F	F	T	F	
F	T	T	T	T	T	
T	F	T	T	F	F	
T	T	F	T	T	T	

Logical Equivalence Example

Are the propositions $p \oplus q$ and f_2: $(p \vee q) \wedge (p \rightarrow q)$ logically equivalent?

p	q	$p \oplus q$	$p \vee q$	$p \rightarrow q$	f_2	$(p \oplus q) \Leftrightarrow f_2$
F	F	F	F	T	F	T
F	T	T	T	T	T	T
T	F	T	T	F	F	F
T	T	F	T	T	T	F

Logical Equivalence Example

Are the propositions $p \oplus q$ and f_2: $(p \vee q) \wedge (p \rightarrow q)$ logically equivalent?

No, they are not equivalent

p	q	$p \oplus q$	$p \vee q$	$p \rightarrow q$	f_2	$(p \oplus q) \Leftrightarrow f_2$
F	F	F	F	T	F	T
F	T	T	T	T	T	T
T	F	T	T	F	F	F
T	T	F	T	T	T	F

Logical Equivalence Examples

Using truth tables we have proven that:

- $(p \rightarrow q) \equiv (q \lor \neg p)$
- $(p \oplus q) \neq f_2$

Exercise

- Prove $(p \oplus q) \equiv \neg(p \Leftrightarrow q)$
- Prove $(p \oplus q) \equiv (p \lor q) \land (p \rightarrow \neg q)$

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- **Equivalence Laws**
- Proving Logical Equivalence

De Morgan's Law

$$(1)\quad \neg(p \land q) \equiv \neg p \lor \neg q$$
$$(2)\quad \neg(p \lor q) \equiv \neg p \land \neg q$$

p	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$
F	F					
F	T					
T	F					
T	T					

Proof for (1). The proof for (2) is similar.

De Morgan's Law

(1) $\neg(p \land q) \equiv \neg p \lor \neg q$
(2) $\neg(p \lor q) \equiv \neg p \land \neg q$

p	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$
F	F	F				
F	T	F				
T	F	F				
T	T	T				

Proof for (1). The proof for (2) is similar.

De Morgan's Law

(1) $\neg(p \land q) \equiv \neg p \lor \neg q$

(2) $\neg(p \lor q) \equiv \neg p \land \neg q$

p	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$
F	F	F	T			
F	T	F	T			
T	F	F	T			
T	T	T	F			

Proof for (1). The proof for (2) is similar.

De Morgan's Law

(1) $\neg(p \land q) \equiv \neg p \lor \neg q$

(2) $\neg(p \lor q) \equiv \neg p \land \neg q$

p	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$
F	F	F	T	T		
F	T	F	T	T		
T	F	F	T	F		
T	T	T	F	F		

Proof for (1). The proof for (2) is similar.

De Morgan's Law

(1) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

(2) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
F	F	F	T	T	T	
F	T	F	T	T	F	
T	F	F	T	F	T	
T	T	T	F	F	F	

Proof for (1). The proof for (2) is similar.

De Morgan's Law

(1) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

(2) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
F	F	F	T	T	T	T
F	T	F	T	T	F	T
T	F	F	T	F	T	T
T	T	T	F	F	F	F

Proof for (1). The proof for (2) is similar.

De Morgan's Law

(1) $\neg(p \land q) \equiv \neg p \lor \neg q$

(2) $\neg(p \lor q) \equiv \neg p \land \neg q$

p	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$
F	F	F	T	T	T	T
F	T	F	T	T	F	T
T	F	F	T	F	T	T
T	T	T	F	F	F	F

Proof for (1). The proof for (2) is similar.

Law of Distributivity

(1) $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

(2) $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

p	*q*	*r*	(q ∧ r)	p ∨ (q ∧ r)	(p ∨ q)	(p ∨ r)	(p ∨ q) ∧ (p ∨ r)
F	F	F	F	**F**	F	F	**F**
F	F	T	F	**F**	F	T	**F**
F	T	F	F	**F**	T	F	**F**
F	T	T	T	**T**	T	T	**T**
T	F	F	F	**T**	T	T	**T**
T	F	T	F	**T**	T	T	**T**
T	T	F	F	**T**	T	T	**T**
T	T	T	T	**T**	T	T	**T**

Examples

Exercise

Prove case (2) of De Morgan's Law the Law of Distributivity using truth tables

<u>De Morgan's Law</u>

$$(2) \quad \neg(p \lor q) \equiv \neg p \land \neg q$$

<u>Law of Distributivity</u>

$$(2) \quad p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$

Law of Contraposition

Law of Contraposition: An implication is always equivalent to its contrapositive

- **Reminder:** the contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
F	F	T	T	T	T
F	T	T	T	F	T
T	F	F	F	T	F
T	T	T	F	F	T

Note: A common proof technique is called proof by contraposition
Prove that the contraposition is true, therefore the implication itself is true.

Converse and Inverse

<u>Converse</u>: The converse of $p \rightarrow q$ is $q \rightarrow p$

<u>Inverse</u>: The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$

Exercises

- Prove that the converse of $p \rightarrow q$ is NOT equivalent to $p \rightarrow q$
- Prove that the inverse of $p \rightarrow q$ is NOT equivalent to $p \rightarrow q$

Logical Equivalence Rules

Equivalence	Name
$p \land T \equiv p$ $\quad$ $p \lor F \equiv p$	Identity laws
$p \lor T \equiv T$ $\quad$ $p \land F \equiv F$	Domination laws
$p \lor p \equiv p$ $\quad$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $\quad$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $\quad$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $\quad$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \lor q) \equiv \neg p \land \neg q$ $\quad$ $\neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $\quad$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv T$ $\quad$ $p \land \neg p \equiv F$	Negation laws

Logical Equivalence Rules

Equivalences with Implication
$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to q) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to q) \lor (q \to r) \equiv (p \land q) \to r$

Equivalences with Bidirectional Implication
$p \Leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \Leftrightarrow q \equiv q \Leftrightarrow p$
$p \Leftrightarrow q \equiv \neg p \Leftrightarrow \neg q$
$p \Leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \Leftrightarrow q) \equiv p \Leftrightarrow \neg q$

Proving Logical Equivalence

- By using equivalence laws, we can prove two propositions are logically equivalent without having to construct large truth tables
- The logical equivalences shown in the tables can be used to construct additional logical equivalences

Proving Logical Equivalence

<u>**Example 1**</u>: Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

Equivalence	Name
$p \land T \equiv p$ $\quad$ $p \lor F \equiv p$	Identity laws
$p \lor T \equiv T$ $\quad$ $p \land F \equiv F$	Domination laws
$p \lor p \equiv p$ $\quad$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $\quad$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $\quad$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $\quad$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \lor q) \equiv \neg p \land \neg q$ $\quad$ $\neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $\quad$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv T$ $\quad$ $p \land \neg p \equiv F$	Negation laws

Proving Logical Equivalence

Example 1: Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

$\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg(\neg p \land q)$ by De Morgan's Law

Equivalence	Name
...	
$\neg(p \lor q) \equiv \neg p \land \neg q$ $\quad$ $\neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's laws
...	

Proving Logical Equivalence

Example 1: Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

$\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg(\neg p \land q)$ by De Morgan's Law

$\equiv \neg p \land (\neg\neg p \lor \neg q)$ by De Morgan's Law

Equivalence	Name
...	
$\neg(p \lor q) \equiv \neg p \land \neg q$ $\neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's laws
...	

Proving Logical Equivalence

Example 1: Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

$\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg(\neg p \land q)$ by De Morgan's Law

$\equiv \neg p \land (\neg\neg p \lor \neg q)$ by De Morgan's Law

$\equiv \neg p \land (p \lor \neg q)$ by Double Negation Law

Equivalence	Name
...	
$\neg(\neg p) \equiv p$	Double negation law
...	

Proving Logical Equivalence

Example 1: Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

$\neg(p \lor (\neg p \land q))$	$\equiv \neg p \land \neg(\neg p \land q)$	by De Morgan's Law
	$\equiv \neg p \land (\neg\neg p \lor \neg q)$	by De Morgan's Law
	$\equiv \neg p \land (p \lor \neg q)$	by Double Negation Law
	$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$	by Distributive Law

Equivalence	Name
…	
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws

Proving Logical Equivalence

Example 1: Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

$\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg(\neg p \land q)$ by De Morgan's Law

$\equiv \neg p \land (\neg\neg p \lor \neg q)$ by De Morgan's Law

$\equiv \neg p \land (p \lor \neg q)$ by Double Negation Law

$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$ by Distributive Law

$\equiv (p \land \neg p) \lor (\neg p \land \neg q)$ by Commutative Law

Equivalence	Name
...	
$p \lor q \equiv q \lor p$ $\quad p \land q \equiv q \land p$	Commutative laws
...	

Proving Logical Equivalence

Example 1: Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

$\neg(p \lor (\neg p \land q))$	$\equiv \neg p \land \neg(\neg p \land q)$	by De Morgan's Law
	$\equiv \neg p \land (\neg\neg p \lor \neg q)$	by De Morgan's Law
	$\equiv \neg p \land (p \lor \neg q)$	by Double Negation Law
	$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$	by Distributive Law
	$\equiv (p \land \neg p) \lor (\neg p \land \neg q)$	by Commutative Law
	$\equiv \mathrm{F} \lor (\neg p \land \neg q)$	by Negation Law

Equivalence	Name
...	
$p \lor \neg p \equiv \mathrm{T}$ $p \land \neg p \equiv \mathrm{F}$	Negation laws
...	

Proving Logical Equivalence

Example 1: Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

$\neg(p \lor (\neg p \land q))$	$\equiv \neg p \land \neg(\neg p \land q)$	by De Morgan's Law
	$\equiv \neg p \land (\neg\neg p \lor \neg q)$	by De Morgan's Law
	$\equiv \neg p \land (p \lor \neg q)$	by Double Negation Law
	$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$	by Distributive Law
	$\equiv (p \land \neg p) \lor (\neg p \land \neg q)$	by Commutative Law
	$\equiv F \lor (\neg p \land \neg q)$	by Negation Law
	$\equiv (\neg p \land \neg q) \lor F$	by Commutative Law

Proving Logical Equivalence

Example 1: Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

$\neg(p \lor (\neg p \land q))$	$\equiv \neg p \land \neg(\neg p \land q)$	by De Morgan's Law
	$\equiv \neg p \land (\neg\neg p \lor \neg q)$	by De Morgan's Law
	$\equiv \neg p \land (p \lor \neg q)$	by Double Negation Law
	$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$	by Distributive Law
	$\equiv (p \land \neg p) \lor (\neg p \land \neg q)$	by Commutative Law
	$\equiv F \lor (\neg p \land \neg q)$	by Negation Law
	$\equiv (\neg p \land \neg q) \lor F$	by Commutative Law
	$\equiv (\neg p \land \neg q)$	by Identity Law

Proving Logical Equivalence

In General:

- Each line should be equivalent to the previous
- Each line should list the law that led to it
    - Exactly one law applied per line
- Start with LHS and go until you reach the RHS

Note: Logical equivalence proofs are very exact. Later proofs will be less restrictive.

Proving Logical Equivalence

Exercise: Prove $(p \land q) \rightarrow (p \lor q) \equiv \mathrm{T}$

Equivalence	Name
$p \land \mathrm{T} \equiv p$ $p \lor \mathrm{F} \equiv p$	Identity laws
$p \lor \mathrm{T} \equiv \mathrm{T}$ $p \land \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \lor q) \equiv \neg p \land \neg q$ $\neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathrm{T}$ $p \land \neg p \equiv \mathrm{F}$	Negation laws

Equivalences with Implication
$p \rightarrow q \equiv \neg p \lor q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \lor q \equiv \neg p \rightarrow q$
$p \land q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \land \neg q$
$(p \rightarrow q) \land (p \rightarrow r) \equiv p \rightarrow (q \land r)$
$(p \rightarrow q) \land (q \rightarrow r) \equiv (p \lor q) \rightarrow r$
$(p \rightarrow q) \lor (p \rightarrow r) \equiv p \rightarrow (q \lor r)$
$(p \rightarrow q) \lor (q \rightarrow r) \equiv (p \land q) \rightarrow r$

Proving Logical Equivalence

Exercise: Prove $(p \wedge q) \rightarrow (p \vee q) \equiv T$

$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ by Conditional Law

Proving Logical Equivalence

Exercise: Prove $(p \land q) \to (p \lor q) \equiv \mathrm{T}$

$(p \land q) \to (p \lor q)$	$\equiv \neg(p \land q) \lor (p \lor q)$	by Conditional Law
	$\equiv (\neg p \lor \neg q) \lor (p \lor q)$	by De Morgan's Law

Proving Logical Equivalence

Exercise: Prove $(p \land q) \rightarrow (p \lor q) \equiv T$

$(p \land q) \rightarrow (p \lor q)$	$\equiv \neg(p \land q) \lor (p \lor q)$	by Conditional Law
	$\equiv (\neg p \lor \neg q) \lor (p \lor q)$	by De Morgan's Law
	$\equiv \neg p \lor (\neg q \lor (p \lor q))$	by Associative Law

Proving Logical Equivalence

Exercise: Prove $(p \land q) \to (p \lor q) \equiv T$

$$
\begin{aligned}
(p \land q) \to (p \lor q) &\equiv \neg(p \land q) \lor (p \lor q) && \text{by Conditional Law} \\
&\equiv (\neg p \lor \neg q) \lor (p \lor q) && \text{by De Morgan's Law} \\
&\equiv \neg p \lor (\neg q \lor (p \lor q)) && \text{by Associative Law} \\
&\equiv \neg p \lor ((\neg q \lor p) \lor q) && \text{by Associative Law}
\end{aligned}
$$

Proving Logical Equivalence

Exercise: Prove $(p \land q) \to (p \lor q) \equiv T$

$(p \land q) \to (p \lor q)$	$\equiv \neg(p \land q) \lor (p \lor q)$	by Conditional Law
	$\equiv (\neg p \lor \neg q) \lor (p \lor q)$	by De Morgan's Law
	$\equiv \neg p \lor (\neg q \lor (p \lor q))$	by Associative Law
	$\equiv \neg p \lor ((\neg q \lor p) \lor q)$	by Associative Law
	$\equiv \neg p \lor ((p \lor \neg q) \lor q)$	by Commutative Law

Proving Logical Equivalence

Exercise: Prove $(p \land q) \rightarrow (p \lor q) \equiv T$

$(p \land q) \rightarrow (p \lor q)$	$\equiv \neg(p \land q) \lor (p \lor q)$	by Conditional Law
	$\equiv (\neg p \lor \neg q) \lor (p \lor q)$	by De Morgan's Law
	$\equiv \neg p \lor (\neg q \lor (p \lor q))$	by Associative Law
	$\equiv \neg p \lor ((\neg q \lor p) \lor q)$	by Associative Law
	$\equiv \neg p \lor ((p \lor \neg q) \lor q)$	by Commutative Law
	$\equiv \neg p \lor (p \lor (\neg q \lor q))$	by Associative Law

Proving Logical Equivalence

Exercise: Prove $(p \land q) \rightarrow (p \lor q) \equiv \mathrm{T}$

$(p \land q) \rightarrow (p \lor q)$	$\equiv \neg(p \land q) \lor (p \lor q)$	by Conditional Law
	$\equiv (\neg p \lor \neg q) \lor (p \lor q)$	by De Morgan's Law
	$\equiv \neg p \lor (\neg q \lor (p \lor q))$	by Associative Law
	$\equiv \neg p \lor ((\neg q \lor p) \lor q)$	by Associative Law
	$\equiv \neg p \lor ((p \lor \neg q) \lor q)$	by Commutative Law
	$\equiv \neg p \lor (p \lor (\neg q \lor q))$	by Associative Law
	$\equiv (\neg p \lor p) \lor (\neg q \lor q)$	by Associative Law

Proving Logical Equivalence

Exercise: Prove $(p \land q) \rightarrow (p \lor q) \equiv T$

$(p \land q) \rightarrow (p \lor q)$	$\equiv \neg(p \land q) \lor (p \lor q)$	by Conditional Law
	…	…
	$\equiv (\neg p \lor p) \lor (\neg q \lor q)$	by Associative Law

Proving Logical Equivalence

Exercise: Prove $(p \land q) \to (p \lor q) \equiv T$

$(p \land q) \to (p \lor q)$	$\equiv \neg(p \land q) \lor (p \lor q)$	by Conditional Law
	…	…
	$\equiv (\neg p \lor p) \lor (\neg q \lor q)$	by Associative Law
	$\equiv (p \lor \neg p) \lor (\neg q \lor q)$	by Commutative Law

Proving Logical Equivalence

Exercise: Prove $(p \land q) \to (p \lor q) \equiv T$

$(p \land q) \to (p \lor q)$	$\equiv \neg(p \land q) \lor (p \lor q)$	by Conditional Law
	…	…
	$\equiv (\neg p \lor p) \lor (\neg q \lor q)$	by Associative Law
	$\equiv (p \lor \neg p) \lor (\neg q \lor q)$	by Commutative Law
	$\equiv (p \lor \neg p) \lor (q \lor \neg q)$	by Commutative Law

Proving Logical Equivalence

Exercise: Prove $(p \wedge q) \to (p \vee q) \equiv \mathrm{T}$

$(p \wedge q) \to (p \vee q)$	$\equiv \neg(p \wedge q) \vee (p \vee q)$	by Conditional Law
	…	…
	$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	by Associative Law
	$\equiv (p \vee \neg p) \vee (\neg q \vee q)$	by Commutative Law
	$\equiv (p \vee \neg p) \vee (q \vee \neg q)$	by Commutative Law
	$\equiv \mathrm{T} \vee (q \vee \neg q)$	by Negation Law

Proving Logical Equivalence

Exercise: Prove $(p \land q) \rightarrow (p \lor q) \equiv \mathrm{T}$

$(p \land q) \rightarrow (p \lor q)$	$\equiv \neg(p \land q) \lor (p \lor q)$	by Conditional Law
	…	…
	$\equiv (\neg p \lor p) \lor (\neg q \lor q)$	by Associative Law
	$\equiv (p \lor \neg p) \lor (\neg q \lor q)$	by Commutative Law
	$\equiv (p \lor \neg p) \lor (q \lor \neg q)$	by Commutative Law
	$\equiv \mathrm{T} \lor (q \lor \neg q)$	by Negation Law
	$\equiv \mathrm{T} \lor \mathrm{T}$	by Negation Law

Proving Logical Equivalence

Exercise: Prove $(p \land q) \to (p \lor q) \equiv \mathrm{T}$

$(p \land q) \to (p \lor q)$	$\equiv \neg(p \land q) \lor (p \lor q)$	by Conditional Law
	…	…
	$\equiv (\neg p \lor p) \lor (\neg q \lor q)$	by Associative Law
	$\equiv (p \lor \neg p) \lor (\neg q \lor q)$	by Commutative Law
	$\equiv (p \lor \neg p) \lor (q \lor \neg q)$	by Commutative Law
	$\equiv \mathrm{T} \lor (q \lor \neg q)$	by Negation Law
	$\equiv \mathrm{T} \lor \mathrm{T}$	by Negation Law
	$\equiv \mathrm{T}$	by Domination Law

Proving Logical Equivalence

Exercise: Prove $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

Note that by the conditional laws, we can rewrite the RHS as:

$$\neg p \vee (q \wedge r)$$

And now, we can start from the LHS and try to get to the new RHS

Proving Logical Equivalence

Exercise: Prove $(p \to q) \land (p \to r) \equiv \neg p \lor (q \land r)$

Equivalence	Name
$p \land T \equiv p$ $p \lor F \equiv p$	Identity laws
$p \lor T \equiv T$ $p \land F \equiv F$	Domination laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \lor q) \equiv \neg p \land \neg q$ $\neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv T$ $p \land \neg p \equiv F$	Negation laws

Equivalences with Implication
$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to q) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to q) \lor (q \to r) \equiv (p \land q) \to r$

Proving Logical Equivalence

Exercise: Prove $(p \to q) \land (p \to r) \equiv \neg p \lor (q \land r)$

$(p \to q) \land (p \to r) \equiv (\neg p \lor q) \land (p \to r)$ by Conditional Law

Proving Logical Equivalence

Exercise: Prove $(p \to q) \land (p \to r) \equiv \neg p \lor (q \land r)$

$(p \to q) \land (p \to r)$ $\equiv (\neg p \lor q) \land (p \to r)$ by Conditional Law

$\equiv (\neg p \lor q) \land (\neg p \lor r)$ by Conditional Law

Proving Logical Equivalence

Exercise: Prove $(p \rightarrow q) \land (p \rightarrow r) \equiv \neg p \lor (q \land r)$

$(p \rightarrow q) \land (p \rightarrow r) \equiv (\neg p \lor q) \land (p \rightarrow r)$ by Conditional Law

$\equiv (\neg p \lor q) \land (\neg p \lor r)$ by Conditional Law

$\equiv \neg p \lor (q \land r)$ by Distributive Law

Propositional Satisfiability

A compound proposition is <u>**satisfiable**</u> if there is an assignment of truth values to its variables that makes it true.

A compound proposition is <u>**unsatisfiable**</u> when no such assignment exists

- A compound proposition is unsatisfiable iff its negation is a tautology
- An assignment of truth values that make a compound proposition true is called a solution to that satisfiability problem

Propositional Satisfiability

Example: $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ is true when ***p***, ***q***, ***r*** have the same truth value.

p = T, ***q*** = T, ***r*** = T is a solution to this satisfiability problem
p = F, ***q*** = F, ***r*** = F is also a solution

A Few Tips...

- Two aspects to getting comfortable with logical equivalence proofs
  - Recognizing the patterns required to be able to apply a particular law
  - Knowing which laws to actually apply
- Both require PRACTICE