CSE 191

Introduction to Discrete Structures

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Predicates and Quantifiers

Recap

- Propositions
  - Declarative statements – either TRUE or FALSE
  - Can represent them with propositional variables
  - Can be combined/modified with logical operators
- Truth tables
  - Lists all possible combinations of truth values for the atomics in a compound proposition and the resulting truth value
- Logical Equivalence
  - Two compound propositions are logically equivalent if they have the same truth value no matter the truth values of their atomics
  - Can be proven with truth tables, or by applying laws of equivalence

Outline

Predicates and Quantifiers

- **From Propositions to Predicates**
- Quantifiers

From Propositions to Predicates

Consider the statement: "X is even"

- Contains a variable, ***X***, so it is not a proposition
  - Given a value for ***X***, we can determine the truth value
  - Once we know the ***X***, the statement is TRUE or FALSE, but not both
- Sentences whose truth value is based on variables are **predicates**

Predicates

A <u>**predicate**</u> is a *function* that takes some *variable(s) as arguments*; it returns either TRUE or FALSE, but never both, depending on the combination of the combination of values passed as arguments.

a proposition can be thought of as a function of 0 variables

Predicates

A **predicate** is a *function* that takes some *variable(s) as arguments*; it returns either TRUE or FALSE, but never both, depending on the combination of the combination of values passed as arguments.

Example: $P(x)$: x is an even number.

P is the function, ***x*** is the variable

P*(*x*)** is the value of the predicate ***P at ***x***

Predicates

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P*(*x*)** is the value of the predicate ***P at ***x***

What are the possible values of X?

Domain of Discourse

Given a predicate, ***P*(*x*)**, the **domain of discourse** (often just called the **domain**) is the set of all possible values for the variable ***x***.

- Predicates with multiple variables may have:
  - Multiple domains of discourse (one for each variable)
  - A single domain of discourse for all variables

Examples

Example 1

Consider the predicate ***Q*(*x*,*y*):** *y* is enrolled in recitation *x*.

- We can define the domain of discourse of ***x*** as {C1, C2, C3}
- …and the domain of discourse of ***y*** as {all students in CSE 191}

Example 2

Consider the predicate ***R*(*x*,*y*):** *x* and *y* are friends.

- We can define the domain of discourse of ***R*** as {all students at UB}

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

P*(*x*,*y*):** $2x = y$, where the domain for ***x is {1,2,3}, ***y*** is {4,5,6}

Statement	Is Proposition?	Truth Value
***P*(1,4): 2(1) = 4**		

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

P*(*x*,*y*):** $2x = y$, where the domain for ***x is {1,2,3}, ***y*** is {4,5,6}

Statement	Is Proposition?	Truth Value
***P*(1,4): 2(1) = 4**	Yes	

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

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Statement	Is Proposition?	Truth Value
***P*(1,4): 2(1) = 4**	Yes	FALSE

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

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Statement	Is Proposition?	Truth Value
***P*(1,4): 2(1) = 4**	Yes	FALSE
***P*(2,4): 2(2) = 4**		

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

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***P*(x,4): 2(x) = 4**		

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***P*(x,4): 2(x) = 4**	No	—

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

P*(*x*,*y*):** 2*x* = *y*, where the domain for ***x is {1,2,3}, ***y*** is {4,5,6}

Statement	Is Proposition?	Truth Value
***P*(1,4): 2(1) = 4**	Yes	FALSE
***P*(2,4): 2(2) = 4**	Yes	TRUE
***P*(x,4): 2(x) = 4**	No	—

What about ***P*(2,3): 2(2) = 3?**

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

P*(*x*,*y*):** $2x = y$, where the domain for ***x is {1,2,3}, ***y*** is {4,5,6}

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***P*(1,4): 2(1) = 4**	Yes	FALSE
***P*(2,4): 2(2) = 4**	Yes	TRUE
***P*(x,4): 2(x) = 4**	No	—

P(2,3) is meaningless (in this example). 3 is not in the domain of ***y***.

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

P*(*x*,*y*):** $2x = y$, where the domain for ***x is {1,2,3}, ***y*** is {4,5,6}

Statement	Is Proposition?	Truth Value
$P(1,4) \vee P(3,6)$		
$P(1,4) \vee \neg P(3,6)$		
$P(2,4) \rightarrow P(2,5)$		
$P(2,4) \wedge P(x,4)$		

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

P*(*x*,*y*):** $2x = y$, where the domain for ***x is {1,2,3}, ***y*** is {4,5,6}

Statement	Is Proposition?	Truth Value
$P(1,4) \lor P(3,6)$	Yes	TRUE
$P(1,4) \lor \neg P(3,6)$		
$P(2,4) \rightarrow P(2,5)$		
$P(2,4) \land P(x,4)$		

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

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$P(1,4) \vee P(3,6)$	Yes	TRUE
$P(1,4) \vee \neg P(3,6)$	Yes	FALSE
$P(2,4) \rightarrow P(2,5)$		
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$P(1,4) \lor P(3,6)$	Yes	TRUE
$P(1,4) \lor \neg P(3,6)$	Yes	FALSE
$P(2,4) \rightarrow P(2,5)$	Yes	FALSE
$P(2,4) \land P(x,4)$		

Examples (Predicate vs Proposition)

Let the predicate ***P*(*x*,*y*)** be defined by:

P*(*x*,*y*):** $2x = y$, where the domain for ***x is {1,2,3}, ***y*** is {4,5,6}

Statement	Is Proposition?	Truth Value
$P(1,4) \lor P(3,6)$	Yes	TRUE
$P(1,4) \lor \neg P(3,6)$	Yes	FALSE
$P(2,4) \to P(2,5)$	Yes	FALSE
$P(2,4) \land P(x,4)$	No	—

More Examples

Let the predicate $Q(x,y)$ be defined by:

$Q(x,y)$: $x + y > 4$, where the domain for x and y is **all integers**

Which of the following are predicates? Propositions?

- $Q(1,2)$
- $Q(x,2)$
- $Q(1000,y)$
- $Q(1000,2)$
- $Q(x,y)$

More Examples

Let the predicate ***Q*(*x*,*y*)** be defined by:

Q*(*x*,*y*):** $x + y > 4$, where the domain for ***x and ***y*** is **all integers**

Which of the following are predicates? Propositions?

- ***Q*(1,2)** proposition
- ***Q*(*x*,2)** predicate
- ***Q*(1000,*y*)** predicate
- ***Q*(1000,2)** proposition
- ***Q*(*x*,*y*)** predicate

Outline

Predicates and Quantifiers

- From Propositions to Predicates
- **Quantifiers**

Quantifiers

Quantification expresses the extent to which a predicate is true over a range of elements. For example, in English: all, some, none, many, few, …

Universal Quantification

Suppose $P(x)$ is a predicate on some domain, D.

The **universal quantification** of $P(x)$ is the *proposition*:

"$P(x)$ is true for all x in the domain of discourse D."

Written as: $\forall x, P(x)$

Read as: "For all x, $P(x)$" or "For every x, $P(x)$"

$\forall x, P(x)$ is TRUE if $P(x)$ is TRUE for *every* x in D.

$\forall x, P(x)$ is FALSE if $P(x)$ is FALSE for *some* x in D.

Universal Quantification Example

$P(x)$: $x + 2 = 5$, where the domain of discourse is {1,2,3}

$\forall x, P(x)$ means: "for all x in {1,2,3}, $x + 2 = 5$

$\forall x, P(x) \equiv P(1) \wedge P(2) \wedge P(3)$

$\equiv (1 + 2 = 5) \wedge (2 + 2 = 5) \wedge (3 + 2 = 5)$

$\equiv F \wedge F \wedge T$

$\therefore \forall x, P(x)$ is FALSE (since **1** + 2 = 5 and **2** + 2 = 5 are FALSE)

Universal Quantification Example

$P(x): x + 2 = 5$, where the domain of discourse is $\{1,2,3\}$

$\forall x, P(x)$ means: "for all x in $\{1,2,3\}$, $x + 2 = 5$

$$
\begin{aligned}
\forall x, P(x) &\equiv P(1) \land P(2) \land P(3) \\
&\equiv (\mathbf{1} + 2 = 5) \land (\mathbf{2} + 2 = 5) \land (\mathbf{3} + 2 = 5) \\
&\equiv F \land F \land T
\end{aligned}
$$

$\therefore \forall x, P(x)$ is FALSE (since $\mathbf{1} + 2 = 5$ and $\mathbf{2} + 2 = 5$ are FALSE)

Note that ∴ denotes "therefore"

Universal Quantification Example

***P*(*x*): *x* + 2 = 5**, where the domain of discourse is {1,2,3}

∀x, *P*(*x*) means: "for all ***x*** in {1,2,3}, ***x*** + 2 = 5

∀x, *P*(*x*) ≡ ***P*(1) ∧ *P*(2) ∧ *P*(3)**

≡ (**1** + 2 = 5) ∧ (**2** + 2 = 5) ∧ (**3** + 2 = 5)

≡ F ∧ F ∧ T

∴ **∀x, *P*(*x*)** is FALSE (since **1** + 2 = 5 and **2** + 2 = 5 are FALSE)

Note that ∴ denotes "therefore"

An input to that causes a universally quantified statement to evaluate to false is called a **counterexample**

Universal Quantification Example

$P(x)$: $x + 2 = 5$, where the domain of discourse is {1,2,3}

$\forall x, P(x)$ means: "for all x in {1,2,3}, $x + 2 = 5$

$$\forall x, P(x) \equiv P(1) \land P(2) \land P(3)$$
$$\equiv (\mathbf{1} + 2 = 5) \land (\mathbf{2} + 2 = 5) \land (\mathbf{3} + 2 = 5)$$
$$\equiv F \land F \land T$$

An input to that causes a universally quantified statement to evaluate to false is called a **counterexample**

$\therefore \forall x, P(x)$ is FALSE (since **1** + 2 = 5 and **2** + 2 = 5 are FALSE)

Note that ∴ denotes "therefore"

What if the domain of discourse was {3}?

Universal Quantification Example

Consider:

$A(x)$: x is even

$B(x)$: $x^2 > 0$

$C(x)$: $x < 2$

where the domain of A, B, C is {0,1,2,3}

True or False?

$\forall x, (C(x) \rightarrow A(x))$

$\forall x, (B(x) \vee C(x))$

Universal Quantification Example

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$A(x)$: x is even

$B(x)$: $x^2 > 0$

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where the domain of A, B, C is $\{0,1,2,3\}$

True or False?

$\forall x, (C(x) \rightarrow A(x))$

FALSE, counterexample is $x = 1$

$\forall x, (B(x) \vee C(x))$

Universal Quantification Example

Consider:

$A(x)$: x is even

$B(x)$: $x^2 > 0$

$C(x)$: $x < 2$

where the domain of A, B, C is $\{0,1,2,3\}$

True or False?

$\forall x, (C(x) \rightarrow A(x))$

FALSE, counterexample is $x = 1$

$\forall x, (B(x) \vee C(x))$

TRUE

x	$B(x)$	$C(x)$
0	F	T
1	T	T
2	T	F
3	T	F

Universal Quantification Example

Consider:

$S(x)$: x is a student in CSE 191

$T(x)$: x is a CSE major

where the domain for S and T is {all students enrolled in CSE 191}

True or False?

$\forall x, S(x)$

$\forall x, T(x)$

$\forall x, (S(x) \rightarrow T(x))$

$\forall x, (T(x) \rightarrow S(x))$

Universal Quantification Example

Consider:

$S(x)$: x is a student in CSE 191

$T(x)$: x is a CSE major

where the domain for S and T is {all students enrolled in CSE 191}

True or False?

$\forall x, S(x)$ TRUE

$\forall x, T(x)$ FALSE

$\forall x, (S(x) \rightarrow T(x))$ FALSE

$\forall x, (T(x) \rightarrow S(x))$ TRUE

Existential Quantification

Suppose ***P***(***x***) is a predicate on some domain, ***D***.

The **<u>existential quantification</u>** of ***P***(***x***) is the *proposition*:

"***P***(***x***) is true for some ***x*** in the domain of discourse ***D***."

Written as: $\exists x, P(x)$

Read as: "There exists an ***x*** such that, ***P(x)***" or "For some ***x***, ***P(x)***"

$\exists x, P(x)$ is TRUE if ***P***(***x***) is TRUE for *<u>some</u>* ***x*** in ***D***.

$\exists x, P(x)$ is FALSE if ***P***(***x***) is FALSE for *<u>every</u>* ***x*** in ***D***.

Existential Quantification Example

$P(x)$: $x + 2 = 5$, where the domain of discourse is {1,2,3}

$\exists x, P(x)$ means: "for some x in {1,2,3}, $x + 2 = 5$

$\exists x, P(x) \equiv P(1) \lor P(2) \lor P(3)$

$\equiv (1 + 2 = 5) \lor (2 + 2 = 5) \lor (3 + 2 = 5)$

$\equiv F \lor F \lor T$

$\therefore \exists x, P(x)$ is TRUE (because **3** + 2 = 5 is TRUE)

Existential Quantification Example

***P*(*x*): *x* + 2 = 5**, where the domain of discourse is {1,2,3}

∃*x, *P*(*x*)** means: "for some ***x in {1,2,3}, ***x*** + 2 = 5

***∃*x, *P*(*x*)** ≡ ***P*(1)** ⋁ ***P*(2)** ⋁ ***P*(3)**

≡ (**1** + 2 = 5) ⋁ (**2** + 2 = 5) ⋁ (**3** + 2 = 5)

≡ F ⋁ F ⋁ T

∴ *∃*x, *P*(*x*) is TRUE (because **3** + 2 = 5 is TRUE)

An input to that causes a predicate to evaluate to true is called a **satisfying assignment**

Existential Quantification Example

Consider:

$A(x): x = 1$

$B(x): x > 5$

$C(x): x < 5$

where the domain of ***A***, ***B***, ***C*** is {1,2,3}

True or False?

$\exists x, (C(x) \rightarrow A(x))$

$\exists x, B(x)$

Existential Quantification Example

Consider:

$A(x)$: $x = 1$

$B(x)$: $x > 5$

$C(x)$: $x < 5$

where the domain of ***A***, ***B***, ***C*** is {1,2,3}

True or False?

$\exists x, (C(x) \rightarrow A(x))$

TRUE, satisfying assignment x=1

$\exists x, B(x)$

FALSE

$P(1) \lor P(2) \lor P(3) = F \lor F \lor F$

$= F$

Note: The existential and universal quantifiers have higher precedence than all logical operators

Existential Quantification Example

Consider:

$S(x)$: x is a student in CSE 191

$T(x)$: x is a CSE major

where the domain for S and T is {all students enrolled in CSE 191}

True or False?

$\exists x, S(x)$	TRUE
$\exists x, T(x)$	TRUE
$\exists x, (S(x) \rightarrow T(x))$	TRUE
$\exists x, (T(x) \rightarrow S(x))$	TRUE

Quantifier Example

Consider:

$P(x)$: $x^2 > 9$

$Q(x)$: $x^2 \geq 0$

where the domain is *all integers*

True or False?

$\forall x, P(x)$

$\exists x, P(x)$

$\forall x, Q(x)$

$\exists x, Q(x)$

Quantifier Example

Consider:

$P(x)$: $x^2 > 9$

$Q(x)$: $x^2 \geq 0$

where the domain is *all integers*

True or False?

$\forall x, P(x)$	FALSE, consider: x = 2
$\exists x, P(x)$	TRUE, consider: x = 5
$\forall x, Q(x)$	TRUE, consider…math
$\exists x, Q(x)$	TRUE

Binding Variables

The occurrence of a variable, ***x***, is said to be **bound** when a quantifier is used on that variable.

The occurrence of a variable, ***x***, is said to be **free** when it is not bound by a quantifier or set to a particular variable.

The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier.

Note: A variable is free if it is outside of the scope of all quantifiers in the formula that specify this variable.

Binding Example

Consider: $\exists x, (x + y = 1)$

- The variable ***x*** is bound by the existential quantifier $\exists x$
- ***y*** is free

Consider: $\forall x, (x < 9) \vee (x > 9)$

- The variable ***x*** is bound in $(x < 9)$ by $\forall x$
- The variable ***x*** is free in $(x > 9)$

Binding Example

Consider: $\exists x, (x + y = 1)$

- The variable x is bound by the existential quantifier $\exists x$
- y is free

Consider: $\forall x, (x < 9) \vee (x > 9)$

- The variable x is bound in $(x < 9)$ by $\forall x$
- The variable x is free in $(x > 9)$

The precedence of $\forall$ and $\exists$ is higher than logical operators

Quantified Statements and English

Consider:

$L(x,y)$: x loves y

where the domain of x is {all students enrolled in CSE 191} and
where the domain of y is {all courses offered by UB CSE}

$$\exists x, (L(x, \text{CSE 191}) \land L(x, \text{CSE 250}))$$

Quantified Statements and English

Consider:

$L(x,y)$: x loves y

where the domain of x is {all students enrolled in CSE 191} and
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$$\exists x, (L(x, \text{CSE 191}) \land L(x, \text{CSE 250}))$$

A CSE 191 student loves both CSE 191 and CSE 250

Quantified Statements and English

Consider:

$L(x,y)$: x loves y

where the domain of x is {all students enrolled in CSE 191} and
where the domain of y is {all courses offered by UB CSE}

$$\exists x \; \exists y \; \forall z, ((x \neq y) \land (L(x, z) \rightarrow L(y, z)))$$

Quantified Statements and English

Consider:

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where the domain of y is {all courses offered by UB CSE}

$$\exists x\ \exists y\ \forall z, ((x \neq y) \land (L(x, z) \rightarrow L(y, z)))$$

There are 2 different students, ***x*** and ***y***, in CSE 191 such that if ***x*** loves a CSE course, then so does ***y***.

Quantified Statements and English

Consider:

$L(x,y)$: x loves y

where the domain of x is {all students enrolled in CSE 191} and
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Every CSE course is loved by some CSE 191 student

Quantified Statements and English

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where the domain of x is {all students enrolled in CSE 191} and
where the domain of y is {all courses offered by UB CSE}

Every CSE course is loved by some CSE 191 student

$$\forall y\ \exists x, L(x, y)$$

Quantified Statements and English

Consider:

$A(x)$: x lives in Amherst
$B(x)$: x is a CSE 191 student
$C(x)$: x has a good GPA
$D(x)$: x majors in CSE

Domain of discourse: all UB students

All UB students are CSE 191 students

Quantified Statements and English

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$$\forall x, B(x)$$

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$$\forall x, (B(x) \rightarrow C(x))$$

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Domain of discourse: all UB students

CSE 191 students not living in Amherst major in CSE

Quantified Statements and English

Consider:

$A(x)$: x lives in Amherst

$B(x)$: x is a CSE 191 student

$C(x)$: x has a good GPA

$D(x)$: x majors in CSE

Domain of discourse: all UB students

CSE 191 students not living in Amherst major in CSE

$$\forall x, ((B(x) \land \neg A(x)) \rightarrow D(x))$$

Quantified Statements and English

Consider:

$A(x)$: x lives in Amherst

$B(x)$: x is a CSE 191 student

$C(x)$: x has a good GPA

$D(x)$: x majors in CSE

Domain of discourse: all UB students

No CSE 191 student lives in Amherst

Quantified Statements and English

Consider:

$A(x)$: x lives in Amherst

$B(x)$: x is a CSE 191 student

$C(x)$: x has a good GPA

$D(x)$: x majors in CSE

Domain of discourse: all UB students

No CSE 191 student lives in Amherst

$$\forall x, (B(x) \rightarrow \neg A(x))$$

Exercise

Translate the following theorems to quantified statements

If $\boldsymbol{x}$ is an even number, then $\boldsymbol{x} + \mathbf{1}$ is odd

Every even number is a multiple of 2

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If ***x*** is an even number, then ***x* + 1** is odd

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First we must define domain/predicates

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If ***x*** is an even number, then ***x* + 1** is odd

Every even number is a multiple of 2

First we must define domain/predicates

- Domain: all integers
- ***P*(*x*):** ***x*** is an even number
- ***Q*(*x*):** ***x*** is an odd number

Exercise

Translate the following theorems to quantified statements

If x is an even number, then $x + 1$ is odd

Every even number is a multiple of 2

First we must define domain/predicates

- Domain: all integers
- $P(x)$: x is an even number
- $Q(x)$: x is an odd number

$$\forall x, (P(x) \rightarrow Q(x + 1))$$

Exercise

Translate the following theorems to quantified statements

If x is an even number, then $x + 1$ is odd

First we must define domain/predicates

- Domain: all integers
- $P(x)$: x is an even number
- $Q(x)$: x is an odd number

$$\forall x, (P(x) \rightarrow Q(x + 1))$$

Every even number is a multiple of 2

Domain and predicates:

- Domain: all integers
- $R(y)$: y is an even number
- $S(y)$: y is a multiple of 2

Exercise

Translate the following theorems to quantified statements

If x is an even number, then $x + 1$ is odd

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- Domain: all integers
- $P(x)$: x is an even number
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Domain and predicates:

- Domain: all integers
- $R(y)$: y is an even number
- $S(y)$: y is a multiple of 2

$$\forall y, (R(y) \rightarrow S(y))$$

Exercise

Translate the following theorems to quantified statements

If ***x*** is an even number, then ***x* + 1** is odd

First we must define domain/predicates

- Domain: all integers
- ***P***(***x***): ***x*** is an even number
- ***Q***(***x***): ***x*** is an odd number

$$\forall x, (P(x) \rightarrow Q(x + 1))$$

Every even number is a multiple of 2

Domain and predicates:

- Domain: all integers
- ***R***(***y***): ***y*** is an even number
- ***S***(***y***): ***y*** is a multiple of 2

$$\forall y, (R(y) \rightarrow S(y))$$

How can we do this differently? With one predicate?

Exercise

Translate the following theorems to quantified statements

If ***x*** is an even number, then ***x* + 1** is odd

First we must define domain/predicates

- Domain: all integers
- ***P*(*x*): *x*** is an even number
- ***Q*(*x*): *x*** is an odd number

$$\forall x, (P(x) \rightarrow Q(x + 1))$$

Every even number is a multiple of 2

Domain and predicates:

- Domain: all even integers
- ***T*(*z*): *z*** is a multiple of 2

$$\forall z, T(z)$$

Quantifier Negation

Consider:

$D(x)$: x majors in CSE

Domain of discourse:

all UB students

Consider the following:

Not every UB student majors in computer science.

Some UB students do not major in computer science.

Quantifier Negation

Consider:

$D(x)$: x majors in CSE

Domain of discourse:

all UB students

Consider the following:

Not every UB student majors in computer science.

$$\neg \forall x, D(x)$$

Some UB students do not major in computer science.

Quantifier Negation

Consider:

$D(x)$: x majors in CSE

Domain of discourse:

all UB students

Consider the following:

Not every UB student majors in computer science.

$$\neg \forall x, D(x)$$

Some UB students do not major in computer science.

$$\exists x, \neg D(x)$$

Quantifier Negation Rule

Quantifier Negation

In general we have, for any predicate ***P*(*x*)**:

$$\neg \forall x, P(x) \equiv \exists x, \neg P(x) \text{ and } \neg \exists x, P(x) \equiv \forall x, \neg P(x)$$

De Morgan's Law for Quantifiers			
Negation	**Equivalent statement**	**Negation is TRUE when…**	**Negation is FALSE when…**
$\neg \exists x, P(x)$	$\forall x, \neg P(x)$	For every x, P(x) is FALSE	There is an x where P(x) is TRUE
$\neg \forall x, P(x)$	$\exists x, \neg P(x)$	There is an x where P(x) is FALSE	P(x) is true for every x

Quantifier Negation Examples

Negate the following statements and simplify

$\exists x, (P(x) \rightarrow \neg Q(x))$

$\forall x, (P(x) \rightarrow \exists y (P(y) \vee Q(y)))$

Quantifier Negation Examples

Negate the following statements and simplify

$\exists x,(P(x) \rightarrow \neg Q(x))$

$\neg \exists x,(P(x) \rightarrow \neg Q(x))$

$\forall x,\neg(P(x) \rightarrow \neg Q(x))$

$\forall x,\neg(\neg P(x) \vee \neg Q(x))$

$\forall x,(\neg\neg P(x) \wedge \neg\neg Q(x))$

$\forall x,(P(x) \wedge Q(x))$

$\forall x, (P(x) \rightarrow \exists y(P(y) \vee Q(y)))$

$\neg \forall x, (P(x) \rightarrow \exists y(P(y) \vee Q(y)))$

$\exists x, \neg(P(x) \rightarrow \exists y(P(y) \vee Q(y)))$

$\exists x, \neg(\neg P(x) \vee \exists y(P(y) \vee Q(y)))$

$\exists x, (\neg\neg P(x) \wedge \neg \exists y(P(y) \vee Q(y)))$

$\exists x, (P(x) \wedge \forall y \neg(P(y) \vee Q(y)))$

Nested Quantifiers

A logical expression with more than one quantifier that bind different variables in the same predicate is said to have **nested quantifiers**.

In order to evaluate them we must consider their *ordering* and *scope*

Nested Quantifiers: Ordering

The order of the quantifiers is important

(unless they are all universal or all existential quantifiers)

Example

Q(***x***,***y***)**: x + y = 0,** where the domain is all real numbers

- $\exists y \forall x Q(x,y)$: there is a real number ***y***, such that for every real number ***x***, ***x*** **+** ***y*** **= 0**.
- $\forall x \exists y Q(x,y)$: for every real number ***x***, there is a real number ***y*** such that ***x*** **+** ***y*** **= 0**.

Nested Quantifiers: Ordering

The order of the quantifiers is important
(unless they are all universal or all existential quantifiers)

Example

$Q(x,y)$: $x + y = 0$, where the domain is all real numbers

FALSE

- **$\exists y \forall x Q(x,y)$:** there is a real number ***y***, such that for every real number ***x***, ***x + y* = 0**.

TRUE

- **$\forall x \exists y Q(x,y)$:** for every real number ***x***, there is a real number ***y*** such that ***x + y* = 0.**

In general, we cannot switch the ordering and guarantee equivalence

Nested Quantifiers: Ordering

Consecutive quantifiers of the same type can be reordered while maintaining equivalence

Consider predicate $Q(i,j,k)$

$$\forall i\,\forall j\,\forall k\,Q(i,j,k) \equiv \forall j\,\forall k\,\forall i\,Q(i,j,k) \equiv \forall j\,\forall i\,\forall k\,Q(i,j,k) \equiv \forall i\,\forall k\,\forall j\,Q(i,j,k)$$

...

$$\exists i\,\exists j\,\exists k\,Q(i,j,k) \equiv \exists j\,\exists k\,\exists i\,Q(i,j,k) \equiv \exists j\,\exists i\,\exists k\,Q(i,j,k) \equiv \exists i\,\exists k\,\exists j\,Q(i,j,k)$$

...

Nested Quantifiers: Ordering

We can simplify consecutive variables with the same quantifier

$$\forall i\, \forall j\, \forall k Q(i,j,k) \equiv \forall i,j,k, Q(i,j,k)$$
$$\exists i\, \exists j\, \exists k Q(i,j,k) \equiv \exists i,\ j,\ k, Q(i,j,k)$$

…but do so carefully and make sure you don't accidentally mix quantifiers

Nested Quantifiers: Scope

The portion of the formula a quantifier covers is called the **scope**

- The scope of the quantifier is the predicate immediately following
- Precedence is just below parenthesis
- Any variable not covered by any quantifier is a free variable

Consider the following formula:

$$\forall i \ \exists j, (P(i,j) \rightarrow \ \forall k, Q(k,j))$$

Nested Quantifiers: Scope

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Consider the following formula:

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The scope of $\forall i$ is the entire formula

Nested Quantifiers: Scope

The portion of the formula a quantifier covers is called the **scope**

- The scope of the quantifier is the predicate immediately following
- Precedence is just below parenthesis
- Any variable not covered by any quantifier is a free variable

Consider the following formula:

$$\forall i\ \exists j, (P(i,j) \rightarrow \forall k, Q(k,j))$$

The scope of $\exists j$ is the entire formula (other than $\forall i$)

Nested Quantifiers: Scope

The portion of the formula a quantifier covers is called the **scope**

- The scope of the quantifier is the predicate immediately following
- Precedence is just below parenthesis
- Any variable not covered by any quantifier is a free variable

Consider the following formula:

$$\forall i\ \exists j, (P(i,j) \rightarrow \boxed{\forall k, Q(k,j)})$$

The scope of $\forall k$ is the limited to $\boldsymbol{Q(k,j)}$

Nested Quantifiers: Scope

The portion of the formula a quantifier covers is called the **scope**

- The scope of the quantifier is the predicate immediately following
- Precedence is just below parenthesis
- Any variable not covered by any quantifier is a free variable

Consider the following formula:

$$\forall i\ \exists j, (P(i,j) \rightarrow \forall k, Q(k,j))$$

Nested Quantifiers: Scope

Quantifiers can move as long as their scope doesn't encompass additional quantifiers of a different type

$$\forall i \ \exists j, (P(i,j) \rightarrow \ \forall k, Q(k,j))$$

vs

$$\forall i \ \exists j \ \forall k, (P(i,j) \rightarrow \ Q(k,j))$$

Nested Quantifiers: Scope

Quantifiers can move as long as their scope doesn't encompass additional quantifiers of a different type

$$\forall i\ \exists j,\ (P(i,j) \rightarrow \forall k,\ Q(k,j))$$

vs

$$\forall i\ \exists j\ \forall k,\ (P(i,j) \rightarrow Q(k,j))$$

Nested Quantifiers: Scope

Quantifiers can move as long as their scope doesn't encompass additional quantifiers of a different type

$$\forall i\ \exists j,\ (P(i,j) \rightarrow \forall k,\ Q(k,j))$$

vs

$$\forall i\ \exists j\ \forall k,\ (P(i,j) \rightarrow Q(k,j))$$

Notice how each qualifier covers the exact same variables in either case

Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables that were originally bound

$$\forall i\ \exists j, (P(i,j) \rightarrow\ \forall k, Q(k,j))$$

vs

$$\forall i\ \exists j, (\ \forall k, P(i,j) \rightarrow\ Q(k,j))$$

Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables that were originally bound

$$\forall i \; \exists j, \; (P(i,j) \rightarrow \; \forall k, \; Q(k,j))$$

vs

$$\forall i \; \exists j, \; (\forall k, P(i,j) \rightarrow \; Q(k,j))$$

Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables that were originally bound

$$\forall i \; \exists j, \; (P(i,j) \rightarrow \; \forall k, \; Q(k,j))$$

vs

$$\forall i \; \exists j, \; (\forall k, P(i,j)) \rightarrow \; Q(k,j))$$

Notice how ***k*** is no longer bound by a quantifier in the second equation

Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables that were originally bound

$$\forall i\ \exists j,\ (P(i,j) \rightarrow \forall k,\ Q(k,j))$$

vs

$$\forall i\ \exists j,\ (\forall k, P(i,j) \rightarrow Q(k,j))$$

Notice how ***k*** is no longer bound by a quantifier in the second equation

These two statements are NOT equivalent. The first is a proposition, and the second is a predicate with a free variable, *k*