CSE 191

Introduction to Discrete Structures

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Propositional Logic

Logic and Proofs

- Logic is the basis of all correct mathematical arguments (proofs)
- Important in all of CS and CEN:
  - Problem solving
  - Software engineering (requirements specification,verification)
  - Databases (relational algebras,SQL)
  - Computer Architecture (logic gates,verification)
  - AI (automated theorem proving,rule-based ML)
  - Security (threat modeling)
  - …

Outline

Propositional Logic

- **Propositions**
- Logical Operators
- Truth Tables

Proposition

Definition: A **proposition** is a declarative statement

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- Must be either **TRUE (T)** or **FALSE (F)**
  - Cannot be both…

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- An opinion of a specific person is a proposition
  - Their opinion would determine the true value

Proposition

Definition: A **<u>proposition</u>** is a declarative statement

- Must be either **TRUE (T)** or **FALSE (F)**
  - Cannot be both…
- An opinion of a specific person is a proposition
  - Their opinion would determine the true value
- The bits 0/1 are used for F/T
  - Digital logic uses 0/1, LOW/HIGH, or OFF/ON
  - Computers use bits and logic gates for **all** computation

Propositional Logic

Examples of Propositions	
Proposition	**Truth Value**
We are in Talbert 107	TRUE
2 + 2 = 7	FALSE
I love cheesecake	FALSE
1 + 1 = 2	TRUE

Propositional Logic

Examples of Non-Propositions

Non-Proposition	Reason
What time is it?	Questions are not declarations
Do your homework.	Also not a declaration…
2+3	Also not a declaration…
$x + 1 = 2$	Neither true nor false; truth depends on x
Wow!	Neither true nor false

Propositions vs Non-Propositions

Propositions

- Declarative statements
- Either TRUE or FALSE
  - Has exactly ONE truth value
  - Can't be both TRUE and FALSE

Non-Propositions

- Questions
- Commands/requests
- Statements with unassigned variables
- Exclamations
- etc

Propositional Variables

Propositional variables are variables that represent propositions

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    - Alternatively, the first letter of what we are trying to represent

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Propositional Variables

Propositional variables are variables that represent propositions

- Commonly used letters are **p**, **q**, **r**, **s**, …
  - Alternatively, the first letter of what we are trying to represent
- May be associated a specific proposition or left as a placeholder for an arbitrary proposition
- **Compound propositions** are formed by using propositional variables and logical operators
  - A compound proposition is itself a proposition

Propositional Variables

The **truth value** of a proposition is TRUE or FALSE

- **T** for true propositions
- **F** for false propositions

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Examples

- ***p***: Chicago is the capital of the USA
- ***q***: Albany is the capital of NYS

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Examples

- $\boxed{\boldsymbol{p}:}$ Chicago is the capital of the USA
- ***q***: Albany is the capital of NYS

We use a ":" to define a proposition.

Propositional Variables

The **truth value** of a proposition is TRUE or FALSE

- **T** for true propositions
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Examples

- ***p***: Chicago is the capital of the USA
- ***q***: Albany is the capital of NYS

We use a ":" to define a proposition.

Now we can ask questions like:

- What is the truth value of ***p***?
- What is the truth value of ***q***?

What about sentences like:

- ***p*** and ***q***?
- ***p*** or ***q***?

Outline

Propositional Logic

- Propositions
- **Logical Operators**
- Truth Tables

Logical Operators

Logical Operators allow combining propositions into new ones

- Going forward: combine propositions to form new ones
- Going backward: decompose proposition into atomics

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Example Compound Proposition

If I am at work, then I am wearing sneakers

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If **I am at work**, then **I am wearing sneakers**

Atomic propositions

Logical Operators

Logical Operators allow combining propositions into new ones

- Going forward: combine propositions to form new ones
- Going backward: decompose proposition into atomics

Example Compound Proposition

If I am at work**, then** I am wearing sneakers

Logical operator (if …, then …)

Negation Operator

Let p be a proposition.

The *negation of* p, denoted by $\neg p$ (or sometimes $\bar{p}$) is the statement:

"It is not the case that p"

- $\neg p$ is a new proposition, read as "not p"
- $\neg$ is referred to as the negation operator. It is a ***unary*** operator
  - Unary operators only operate on one proposition
- The truth value of $\neg p$ is the opposite of the truth value of p

Negation Operator Example

Let ***p*** be the following proposition:

p: CSE116 is a prerequisite for CSE191

Negation Operator Example

Let $\boldsymbol{p}$ be the following proposition:

$\boldsymbol{p}$: CSE116 is a prerequisite for CSE191

The negation of $\boldsymbol{p}$ is therefore:

$\boldsymbol{\neg p}$: It is not the case that CSE116 is a prerequisite for CSE191

Negation Operator Example

Let ***p*** be the following proposition:

p: CSE116 is a prerequisite for CSE191

The negation of ***p*** is therefore:

$\neg$***p***: It is not the case that CSE116 is a prerequisite for CSE191

Or more simply:

$\neg$***p***: CSE116 is not a prerequisite for CSE191

Negation Operator Example

Let p be the following proposition:

p: CSE116 is a prerequisite for CSE191

The negation of p is therefore:

$\neg p$: It is not the case that CSE116 is a prerequisite for CSE191

Or more simply:

$\neg p$: CSE116 is not a prerequisite for CSE191

In this case p is TRUE, therefore $\neg p$ is FALSE

Binary Logical Operators

What other connectives do we have in English?

Binary Logical Operators

What other connectives do we have in English?

… and …

… or …

If …, then …

… if and only if …

Unary vs Binary

Unary Operators

- Transform **one proposition** into another
- ie: $\neg p$

Binary Operators

- Combine **two propositions** into one **compound proposition**
- ie: ***p*** and ***q***, ***p*** or ***q***, if ***p***, then ***q***, etc…

Binary Logical Operators: Conjunction

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be propositions.

The *conjunction of* $\boldsymbol{p}$ and $\boldsymbol{q}$, denoted by $\boldsymbol{p} \land \boldsymbol{q}$ is the statement:

"$\boldsymbol{p}$ and $\boldsymbol{q}$"

and is only TRUE when $\boldsymbol{p}$ and $\boldsymbol{q}$ are both TRUE, and is FALSE otherwise

Binary Logical Operators: Conjunction

r: It is rainy *and* windy [conjunction of two propositions]

Binary Logical Operators: Conjunction

r: It is rainy *and* windy   [conjunction of two propositions]

- ***p***: it is rainy
- ***q***: it is windy
- $\boldsymbol{p \land q}$: it is rainy and it is windy

Binary Logical Operators: Conjunction

r: It is rainy *and* windy [conjunction of two propositions]

- ***p***: it is rainy
- ***q***: it is windy
- ***p* ∧ *q***: it is rainy and it is windy
    - Simplified: ***p* ∧ *q***: it is rainy and windy
    - ***p* ∧ *q*** and ***r*** are interchangeable

<u>NOTE:</u> When converting to English, try to use the most natural wording

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What if today is rainy, but not windy?

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- ***p***: it is rainy
- ***q***: it is windy
- $p \land q$: it is rainy and it is windy
  - Simplified: $p \land q$: it is rainy and windy
  - $p \land q$ and ***r*** are interchangeable

NOTE: When converting to English, try to use the most natural wording

What if today is rainy, but not windy?

p is TRUE, ***q*** is FALSE, therefore $p \land q = T \land F = \text{FALSE}$

Binary Logical Operators: Disjunction

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be propositions.

The *disjunction of* $\boldsymbol{p}$ and $\boldsymbol{q}$, denoted by $\boldsymbol{p} \vee \boldsymbol{q}$ is the statement:

"$\boldsymbol{p}$ or $\boldsymbol{q}$"

and is TRUE when $\boldsymbol{p}$ is TRUE, $\boldsymbol{q}$ is TRUE, or both are TRUE

$\boldsymbol{p} \vee \boldsymbol{q}$ is FALSE only when both $\boldsymbol{p}$ and $\boldsymbol{q}$ are FALSE

Binary Logical Operators: Disjunction

r: the playing card is a Queen or a Heart   [disjunction of two propositions]

Binary Logical Operators: Disjunction

r: the playing card is a Queen or a Heart [disjunction of two propositions]

- ***p***: the playing card is a Queen
- ***q***: the playing card is a Heart
- $\boldsymbol{p \vee q}$: the playing card is a Queen or the playing card is a Heart
    - Simplified: $\boldsymbol{p \vee q}$ the playing card is a Queen or a Heart
    - ***r*** and $\boldsymbol{p \vee q}$ are interchangeable

Binary Logical Operators: Disjunction

What is the value of ***r*** if the playing card in question is:

The Ace of Spades?

Binary Logical Operators: Disjunction

What is the value of ***r*** if the playing card in question is:

The Ace of Spades? ***p*** is FALSE, ***q*** is FALSE, so ***p*** ∨ ***q*** = F ∨ F = FALSE

Binary Logical Operators: Disjunction

What is the value of ***r*** if the playing card in question is:

The Ace of Spades? ***p*** is FALSE, ***q*** is FALSE, so ***p*** $\lor$ ***q*** = F $\lor$ F = FALSE

The 2 of Hearts?

Binary Logical Operators: Disjunction

What is the value of $\boldsymbol{r}$ if the playing card in question is:

The Ace of Spades? $\boldsymbol{p}$ is FALSE, $\boldsymbol{q}$ is FALSE, so $\boldsymbol{p} \lor \boldsymbol{q}$ = F $\lor$ F = FALSE

The 2 of Hearts? $\boldsymbol{p}$ is FALSE, $\boldsymbol{q}$ is TRUE, so $\boldsymbol{p} \lor \boldsymbol{q}$ = F $\lor$ T = TRUE

Binary Logical Operators: Disjunction

What is the value of ***r*** if the playing card in question is:

The Ace of Spades? ***p*** is FALSE, ***q*** is FALSE, so ***p*** ∨ ***q*** = F ∨ F = FALSE

The 2 of Hearts? ***p*** is FALSE, ***q*** is TRUE, so ***p*** ∨ ***q*** = F ∨ T = TRUE

The Queen of Clubs?

Binary Logical Operators: Disjunction

What is the value of ***r*** if the playing card in question is:

The Ace of Spades? ***p*** is FALSE, ***q*** is FALSE, so ***p*** $\vee$ ***q*** = F $\vee$ F = FALSE

The 2 of Hearts? ***p*** is FALSE, ***q*** is TRUE, so ***p*** $\vee$ ***q*** = F $\vee$ T = TRUE

The Queen of Clubs? ***p*** is TRUE, ***q*** is FALSE, so ***p*** $\vee$ ***q*** = T $\vee$ F = TRUE

Binary Logical Operators: Disjunction

What is the value of $\boldsymbol{r}$ if the playing card in question is:

The Ace of Spades? $\boldsymbol{p}$ is FALSE, $\boldsymbol{q}$ is FALSE, so $\boldsymbol{p} \lor \boldsymbol{q}$ = F $\lor$ F = FALSE

The 2 of Hearts? $\boldsymbol{p}$ is FALSE, $\boldsymbol{q}$ is TRUE, so $\boldsymbol{p} \lor \boldsymbol{q}$ = F $\lor$ T = TRUE

The Queen of Clubs? $\boldsymbol{p}$ is TRUE, $\boldsymbol{q}$ is FALSE, so $\boldsymbol{p} \lor \boldsymbol{q}$ = T $\lor$ F = TRUE

The Queen of Hearts?

Binary Logical Operators: Disjunction

What is the value of ***r*** if the playing card in question is:

The Ace of Spades? ***p*** is FALSE, ***q*** is FALSE, so ***p*** ⋁ ***q*** = F ⋁ F = FALSE

The 2 of Hearts? ***p*** is FALSE, ***q*** is TRUE, so ***p*** ⋁ ***q*** = F ⋁ T = TRUE

The Queen of Clubs? ***p*** is TRUE, ***q*** is FALSE, so ***p*** ⋁ ***q*** = T ⋁ F = TRUE

The Queen of Hearts? ***p*** is TRUE, ***q*** is TRUE, so ***p*** ⋁ ***q*** = T ⋁ T = TRUE

Binary Logical Operators: Disjunction

What is the value of ***r*** if the playing card in question is:

The Ace of Spades? ***p*** is FALSE, ***q*** is FALSE, so $p \lor q$ = F $\lor$ F = FALSE

The 2 of Hearts? ***p*** is FALSE, ***q*** is TRUE, so $p \lor q$ = F $\lor$ T = TRUE

The Queen of Clubs? ***p*** is TRUE, ***q*** is FALSE, so $p \lor q$ = T $\lor$ F = TRUE

The Queen of Hearts? ***p*** is TRUE, ***q*** is TRUE, so $p \lor q$ = T $\lor$ T = TRUE

Note: This is referred to as inclusive or

Binary Logical Operators: Exclusive Or

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be propositions.

The *exclusive or of* $\boldsymbol{p}$ and $\boldsymbol{q}$, denoted by $\boldsymbol{p} \oplus \boldsymbol{q}$ (read XOR) is the statement:

"$\boldsymbol{p}$ or $\boldsymbol{q}$, but not both"

and is TRUE when exactly one of $\boldsymbol{p}$ and $\boldsymbol{q}$ is TRUE, and FALSE otherwise

Binary Logical Operators: Exclusive Or

r: the playing card is a Queen or a Heart (but not both)

- ***p***: the playing card is a Queen
- ***q***: the playing card is a Heart
- $p \oplus q$: the playing card is a Queen or the playing card is a Heart

Binary Logical Operators: Exclusive Or

What is the value of ***r*** if the playing card in question is:

The Ace of Spades?

Binary Logical Operators: Exclusive Or

What is the value of ***r*** if the playing card in question is:

The Ace of Spades? ***p*** is FALSE, ***q*** is FALSE, so $\boldsymbol{p} \oplus \boldsymbol{q}$ = F ⊕ F = FALSE

Binary Logical Operators: Exclusive Or

What is the value of ***r*** if the playing card in question is:

The Ace of Spades? ***p*** is FALSE, ***q*** is FALSE, so ***p*** ⊕ ***q*** = F ⊕ F = FALSE

The 2 of Hearts? ***p*** is FALSE, ***q*** is TRUE, so ***p*** ⊕ ***q*** = F ⊕ T = TRUE

The Queen of Clubs? ***p*** is TRUE, ***q*** is FALSE, so ***p*** ⊕ ***q*** = T ⊕ F = TRUE

Binary Logical Operators: Exclusive Or

What is the value of ***r*** if the playing card in question is:

The Ace of Spades? ***p*** is FALSE, ***q*** is FALSE, so $\boldsymbol{p} \oplus \boldsymbol{q}$ = F ⊕ F = FALSE

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The Queen of Clubs? ***p*** is TRUE, ***q*** is FALSE, so $\boldsymbol{p} \oplus \boldsymbol{q}$ = T ⊕ F = TRUE

The Queen of Hearts? ***p*** is TRUE, ***q*** is TRUE, so $\boldsymbol{p} \oplus \boldsymbol{q}$ = T ⊕ T = **<u>FALSE</u>**

Binary Logical Operators: Implication

Let p and q be propositions.

The *implication of* p on q, denoted by $p \to q$ is the statement:

"p implies q" or "if p, then q"

and is FALSE when p is TRUE, q is FALSE, and TRUE otherwise

p is called the <u>hypothesis</u> or <u>antecedent</u> or <u>precedent</u>

q is called the <u>conclusion</u> or <u>consequence</u>

Binary Logical Operators: Implication

r: If I'm at work, then I'm wearing sneakers

- ***p***: I'm at work
- ***q***: I'm wearing sneakers
- $\boldsymbol{p} \rightarrow \boldsymbol{q}$: If I'm at work, then I'm wearing sneakers
  - Sometimes called a conditional statement

Binary Logical Operators: Implication

What if I'm at work but I'm not wearing sneakers?

Binary Logical Operators: Implication

What if I'm at work but I'm not wearing sneakers?

$\boldsymbol{p}$ is TRUE, $\boldsymbol{q}$ is FALSE, $\boldsymbol{p} \rightarrow \boldsymbol{q}$ is T $\rightarrow$ F, which is FALSE

Binary Logical Operators: Implication

What if I'm at work but I'm not wearing sneakers?

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Binary Logical Operators: Implication

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$\boldsymbol{p}$ is TRUE, $\boldsymbol{q}$ is FALSE, $\boldsymbol{p} \rightarrow \boldsymbol{q}$ is T $\rightarrow$ F, which is FALSE

What if I'm not at work but I'm wearing sneakers?

$\boldsymbol{p}$ is FALSE, $\boldsymbol{q}$ is TRUE, $\boldsymbol{p} \rightarrow \boldsymbol{q}$ is F $\rightarrow$ T, which is TRUE

If the hypothesis is FALSE, we know nothing about the conclusion

Terminology for Implication

Implication statements can be expressed in many ways.

Some common expressions of $p \rightarrow q$:

- if p, then q
- q if p
- q when p
- q unless not p
- p implies q
- p only if q
- q whenever p
- p is sufficient for q
- q is necessary for p
- q follows from p

Converse, Contrapositive, and Inverse

Converse of $p \rightarrow q$: $q \rightarrow p$

Contrapositive of $p \rightarrow q$: $\neg q \rightarrow \neg p$

Inverse of $p \rightarrow q$: $\neg p \rightarrow \neg q$

Converse, Contrapositive, and Inverse

Example

- ***r*: *p* → *q***: If I am drawing, then I am happy
  - ***p*:** I am drawing, ***q***: I am happy

Converse, Contrapositive, and Inverse

Example

- ***r***: $p \rightarrow q$: If I am drawing, then I am happy
  - ***p*:** I am drawing, ***q***: I am happy
- **Converse:**

Converse, Contrapositive, and Inverse

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- r: $p \rightarrow q$: If I am drawing, then I am happy
  - p: I am drawing, q: I am happy
- **Converse:** If I am happy, then I am drawing

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Example

- ***r*: *p* → *q***: If I am drawing, then I am happy
  - ***p*:** I am drawing, ***q***: I am happy
- **Converse:** If I am happy, then I am drawing
- **Contrapositive:**

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Converse, Contrapositive, and Inverse

Example

- $\boldsymbol{r}$**:** $\boldsymbol{p} \rightarrow \boldsymbol{q}$: If I am drawing, then I am happy
  - $\boldsymbol{p}$**:** I am drawing, $\boldsymbol{q}$: I am happy
- **Converse:** If I am happy, then I am drawing
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- ***r*: *p* → *q***: If I am drawing, then I am happy
  - ***p*:** I am drawing, ***q***: I am happy
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Example

- ***r***: $p \rightarrow q$: If I am drawing, then I am happy
  - ***p*****:** I am drawing, ***q***: I am happy
- **Converse:** If I am happy, then I am drawing
- **Contrapositive:** If I am not happy, then I am not drawing
- **Inverse:** If I am not drawing, then I am not happy

Which of these three is equivalent to ***r****?*

(equivalent propositions have the same truth value)

Converse, Contrapositive, and Inverse

Example

- r: $p \to q$: If I am drawing, then I am happy
    - p: I am drawing, q: I am happy
- **Converse:** If I am happy, then I am drawing
- **Contrapositive:** If I am not happy, then I am not drawing
- **Inverse:** If I am not drawing, then I am not happy

*Which of these three is equivalent to **r**?*

(equivalent propositions have the same truth value)

Binary Logical Operators: Bidirectional Implication

Let ***p*** and ***q*** be propositions.

The *bidirectional implication of* ***p*** on ***q***, denoted by ***p*** ⇔ ***q*** is the statement:

"***p*** if and only if ***q***"

and is only TRUE when ***p*** and ***q*** have same truth value, FALSE otherwise

Binary Logical Operators: Bidirectional Implication

Example

- You can take the flight if and only if you buy a ticket
  - $\boldsymbol{p}$: you can take the flight
  - $\boldsymbol{q}$: you buy a ticket
  - $\boldsymbol{p} \Leftrightarrow \boldsymbol{q}$: you can take the flight if and only if you buy a ticket

Terminology for Bidirectional Implication

Common expressions for $p \Leftrightarrow q$:

- ***p*** if and only if ***q***
- ***p*** is necessary and sufficient for ***q***
- if ***p*** then ***q***, and conversely
- ***p*** iff ***q***

Outline

Propositional Logic

- Propositions
- Logical Operators
- **Truth Tables**

Truth Tables

How can we formally specify the behavior of an operator?

How can we show the results of applying an operator to one or more propositions?

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How can we formally specify the behavior of an operator?

How can we show the results of applying an operator to one or more propositions?

Truth Tables!

Truth Table

A **Truth Table** lists all possible combinations of truth values of the operands, as well as the resulting truth value in the rightmost column

Truth Tables: Negation Operator

- The negation operator has a single operand
  - This operand can either be TRUE or FALSE
- The truth value of $\neg p$ is the opposite of the truth value of p

p	$\neg p$
F	T
T	F

Truth table for negation

Truth Tables: Binary Logical Operators

p	q	$p \land q$
F	F	F
F	T	F
T	F	F
T	T	T

Conjunction/AND

Truth Tables: Binary Logical Operators

p	q	$p \lor q$
F	F	F
F	T	T
T	F	T
T	T	T

Disjunction/OR

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

Exclusive Or/XOR

Truth Tables: Binary Logical Operators

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Implication/If …, then …

p	q	$p \Leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Bidirectional Implication/IFF

How do we construct a truth table?

- Need 2^n rows, where n is the number of propositional variables
  - For $\neg \boldsymbol{p}$ we have 1 variable, therefore $2^1 = 2$ rows
  - For $\boldsymbol{p} \vee \boldsymbol{q}$ we have 2 variables so we need $2^2 = 4$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p} \vee \boldsymbol{q}$

Disjunction/OR

How do we construct a truth table?

- We need a row for **every** possible combination of truth values
  - Fill the first half of the first column with F, second half with T

p	q	$p \vee q$
F		
F		
T		
T		

Disjunction/OR

How do we construct a truth table?

- We need a row for **every** possible combination of truth values
  - Fill the first half of the first column with F, second half with T
  - For the second column: fill the first half of each *group* of rows with F, second half with T
  - …continue for additional columns as needed (other than the last column)

p	q	$p \vee q$
F	F	
F	T	
T	F	
T	T	

Disjunction/OR

How do we construct a truth table?

- We need a row for **every** possible combination of truth values
  - Fill the first half of the first column with F, second half with T
  - For the second column: fill the first half of each *group* of rows with F, second half with T
  - …continue for additional columns as needed (other than the last column)
- Determine the truth value of the new proposition in the last column

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Disjunction/OR

How do we construct a truth table?

What if we have more than 2 variables? ie: $\boldsymbol{p \lor q \lor r}$

How do we construct a truth table?

What if we have more than 2 variables? ie: $\boldsymbol{p \lor q \lor r}$

- 3 variables = 2^3 = 8 rows
- (optional) Add additional columns to handle partial results
    - In this case, we can evaluate it as $(\boldsymbol{p} \lor \boldsymbol{q}) \lor \boldsymbol{r}$

p	q	r	$p \lor q$	$p \lor q \lor r$

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p	q	r	$p \lor q$	$p \lor q \lor r$
F				
F				
F				
F				
T				
T				
T				
T				

How do we construct a truth table?

*What if we have more than 2 variables? ie: **p** ∨ **q** ∨ **r***

- 3 variables = 2^3 = 8 rows
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  - In this case, we can evaluate it as (***p*** ∨ ***q***) ∨ ***r***

p	***q***	***r***	***p*** ∨ ***q***	***p*** ∨ ***q*** ∨ ***r***
F	F			
F	F			
F	T			
F	T			
T	F			
T	F			
T	T			
T	T			

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What if we have more than 2 variables? ie: $p \lor q \lor r$

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    - In this case, we can evaluate it as $(p \lor q) \lor r$

p	q	r	$p \lor q$	$p \lor q \lor r$
F	F	F		
F	F	T		
F	T	F		
F	T	T		
T	F	F		
T	F	T		
T	T	F		
T	T	T		

How do we construct a truth table?

What if we have more than 2 variables? ie: $p \lor q \lor r$

- 3 variables = 2^3 = 8 rows
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F	F	T	F	T
F	T	F	T	T
F	T	T	T	T
T	F	F	T	T
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T

Compound Propositions

A **compound proposition** is created by using one or more logical operators

Suppose ***p*** and ***q*** are propositions.

- Compound proposition: $(\boldsymbol{p} \lor \boldsymbol{q}) \land \neg(\boldsymbol{p} \land \boldsymbol{q})$
- This new proposition is formed using AND, OR, and NOT

Let ***p*:** Jim eats pie, ***q*:** Jim eats cake. What is the above proposition in natural language (ie English)?

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r: Jim eats pie or cake but Jim doesn't eat pie and cake

How to construct compound propositions?

Consider how we built ***r*** from ***p*** and ***q***

1. Identify atomic propositions
   p: Jim eats pie, ***q***: Jim eats cake

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Consider how we built $\boldsymbol{r}$ from $\boldsymbol{p}$ and $\boldsymbol{q}$

1. Identify atomic propositions
   $\boldsymbol{p}$: Jim eats pie, $\boldsymbol{q}$: Jim eats cake
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   $(\boldsymbol{p} \vee \boldsymbol{q})$: Jim eats pie or cake

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3. Negate the second piece
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4. Finally, put it all together

$(p \lor q) \land \neg(p \land q)$: Jim eats pie or cake but Jim doesn't eat pie and cake

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Notice how AND becomes but…

$(\boldsymbol{p} \vee \boldsymbol{q}) \wedge \neg(\boldsymbol{p} \wedge \boldsymbol{q})$: Jim eats pie or cake but Jim doesn't eat pie and cake

Examples

Let $\boldsymbol{p}$: The window is closed; $\boldsymbol{q}$: It is raining; $\boldsymbol{r}$: I will run the air conditioner

1. $\neg \boldsymbol{p}$: ???
2. $\boldsymbol{p} \lor \neg \boldsymbol{q}$: ???
3. It is raining but the window is not closed: ???
4. If it is not raining then the window is open: ???
5. $\boldsymbol{p} \Leftrightarrow \boldsymbol{q}$: ???
6. $\boldsymbol{q} \land \neg \boldsymbol{p} \rightarrow \neg \boldsymbol{r}$: ???

Examples

Let p: The window is closed; q: It is raining; r: I will run the air conditioner

1. $\neg p$: The window is open
2. $p \lor \neg q$: The window is closed or it isn't raining
3. It is raining but the window is not closed: $q \land \neg p$
4. If it is not raining then the window is open: $\neg q \rightarrow \neg p$
5. $p \Leftrightarrow q$: The window is closed if and only if it is raining
6. $q \land \neg p \rightarrow \neg r$: If it is raining but the window not closed, then I will not run the AC

How to evaluate compound propositions?

Remember: Each logical operator creates a new proposition

- The outcome is a new proposition
- Therefore the outcome must be TRUE or FALSE

We have two ways to view a compound proposition

- Start with the smaller propositions and build up to the larger one
- Start with the larger proposition and decompose

How to evaluate compound propositions?

Let's say we want to know if "***r*:** Jim eats pie or cake but Jim doesn't eat pie and cake" is true.

Scenario 1: Let's say we know:

- Jim eats cake and Jim eats pie are both true.
- We can evaluate ***r*** from the ground up as a function of ***p*** and ***q***

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Scenario 1: Let's say we know:

- Jim eats cake and Jim eats pie are both true.
- We can evaluate ***r*** from the ground up as a function of ***p*** and ***q***

We have that p = T, q = T

Therefore:
($p \lor q$) and ($p \land q$) are both T

$\neg(p \land q)$ is F

Finally, $(p \lor q) \land \neg(p \land q)$ is T $\land$ F = F

Therefore ***r*** is FALSE.

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*What if we don't know anything about the atomics that form **r**?*

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Consider the general case, AKA build the truth table

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p	q	$(p \vee q)$	$(p \wedge q)$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$

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F	T	T	F	T	T
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Scenario 2: ***r*** is TRUE and ***p*** is TRUE. What is the truth value of ***q*?**

How to evaluate compound propositions?

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F	F	F	F	T	F
F	T	T	F	T	T
T	F	T	F	T	T
T	T	T	T	F	F

Scenario 2: ***r*** is TRUE and ***p*** is TRUE. What is the truth value of ***q*?** FALSE

Operator Precedence

Operator	Precedence
()	0
$\neg$	1
$\wedge$	2
$\vee$	3
$\oplus$	4
$\rightarrow$	5
$\Leftrightarrow$	6

Examples

$\neg p \wedge q$ means $(\neg p) \wedge q$

$p \wedge q \rightarrow r$ means $(p \wedge q) \rightarrow r$

$p \vee q \wedge r \Leftrightarrow p \rightarrow q \oplus r$

means

$(p \vee (q \wedge r)) \Leftrightarrow (p \rightarrow (q \oplus r))$

When in doubt, use parenthesis!