CSE 191

Introduction to Discrete Structures

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Functions and Relations

Outline

- **Binary Relations**
  - **Intro**
  - Partial Ordering
  - Equivalence Relations
- Functions

Binary Relations

A binary relation is a formal way to related two objects, for example:

- Student ***s*** is related to a course ***c*** iff student ***s*** is enrolled in course ***c***
  - Defines a relation between students at UB and course at UB
- Topic ***t*** is related to topic ***s*** iff you need to learn ***s*** before you learn ***t***
  - Reading a chapter ***t*** in the textbook related to ***s*** requires reading ***s*** first
- ***x*** and ***y*** are related iff they share a common divisor

Binary Relations

A **binary relation** between two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is any set $\boldsymbol{R} \subseteq \boldsymbol{A} \times \boldsymbol{B}$

A binary relation **from** $\boldsymbol{A}$ **to** $\boldsymbol{B}$ is a set $\boldsymbol{R}$ of ordered pairs, where the first element of each ordered pair comes from $\boldsymbol{A}$ and the second from $\boldsymbol{B}$

- For any $\boldsymbol{a} \in \boldsymbol{A}$ and $\boldsymbol{b} \in \boldsymbol{B}$ we say that $\boldsymbol{a}$ is related to $\boldsymbol{b}$ iff $(\boldsymbol{a},\boldsymbol{b}) \in \boldsymbol{R}$
- Denoted by $\boldsymbol{a\ R\ b}$

Note: a relation is a binary predicate **R(*a*,*b*):** "***a*** is related to ***b***"

Example

Consider the set of student, ***S* = { Alice, Bob, Carol, Don }**,
and the set of courses, ***C* = { CSE115, CSE116, CSE191 }**

Alice, Bob, and **Carol** are enrolled in **CSE115**

Don is enrolled in **CSE116**

Alice and **Don** are enrolled in **CSE191**

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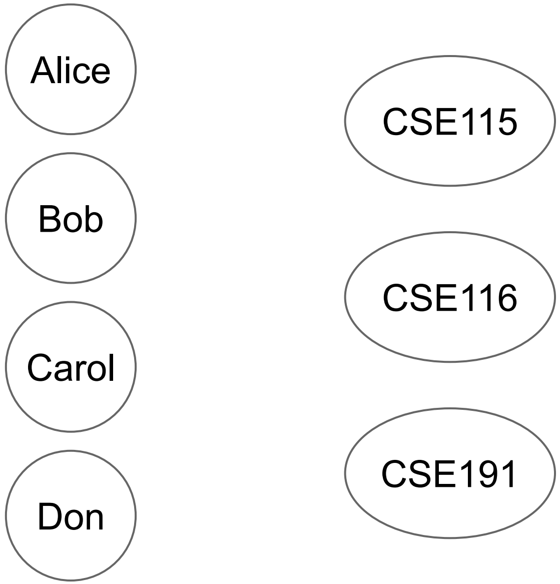

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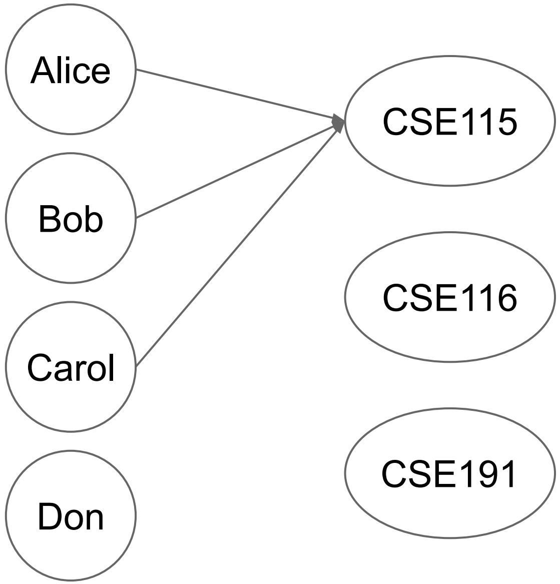

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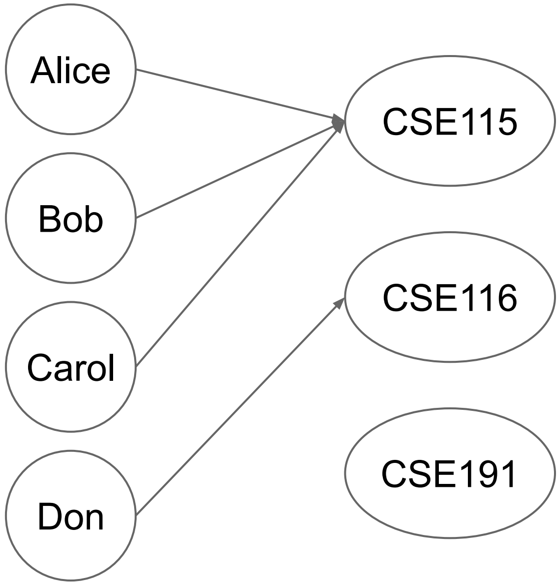

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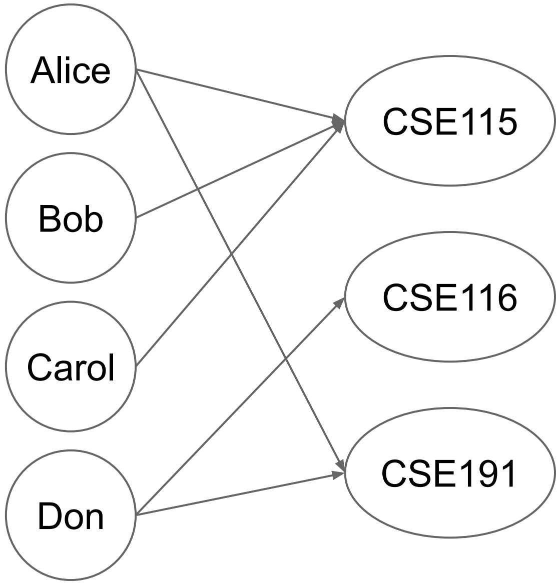

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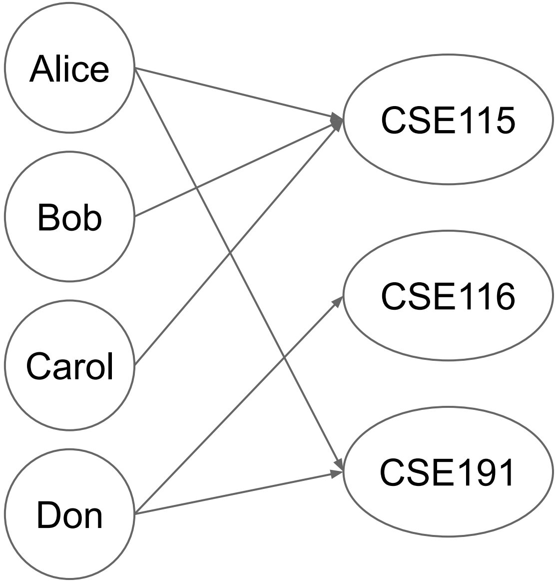
This is called an arrow diagram. It is a visual representation of a binary relation.



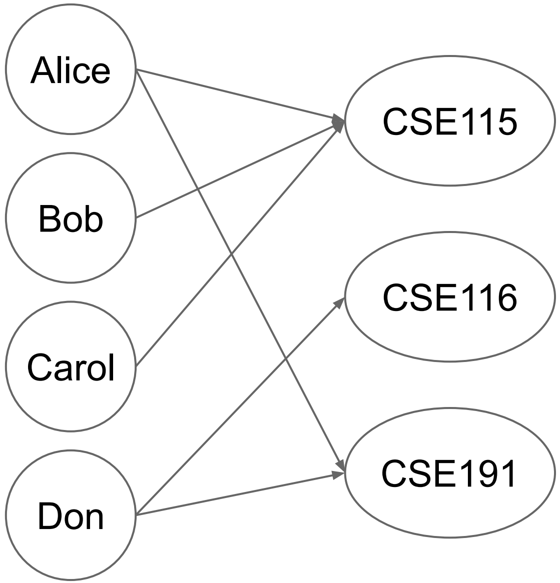
Example

Consider the set of student, ***S* = { Alice, Bob, Carol, Don }**,

and the set of courses, ***C* = { CSE115, CSE116, CSE191 }**

Given the arrow diagram, we have the binary relation ***E*:**

E **= {(Alica, CSE115), (Alice, CSE191), (Bob,CSE115), (Carol,CSE115), (Don,CSE116), (Don,CSE191)}**

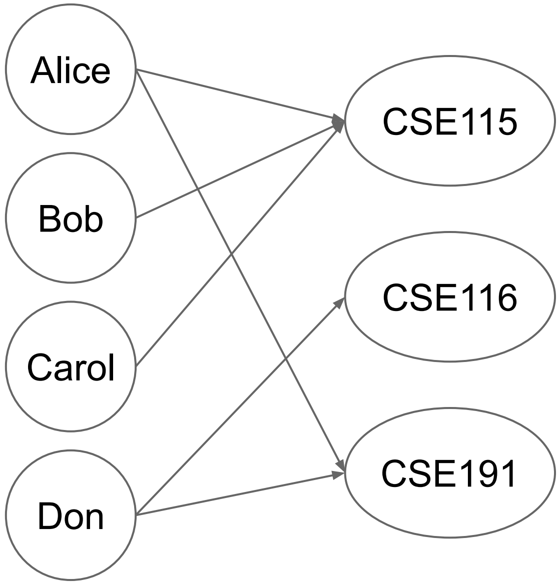


Example

Consider the set of student, ***S* = { Alice, Bob, Carol, Don }**,
and the set of courses, ***C* = { CSE115, CSE116, CSE191 }**

We can also use **matrix representation** to describe ***E***:

	CSE115	CSE116	CSE191
Alice	1	0	1
Bob	1	0	0
Carol	1	0	0
Don	0	1	1



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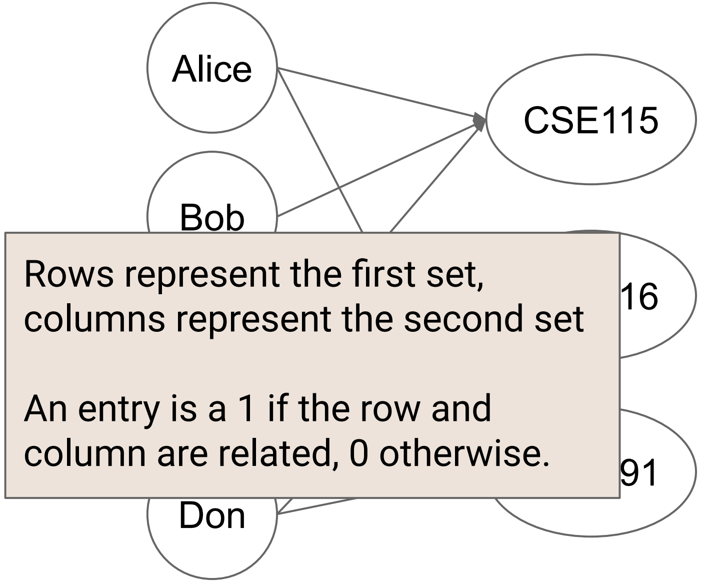


Rows represent the first set, columns represent the second set

An entry is a 1 if the row and column are related, 0 otherwise.

Binary Relations over Infinite Sets

Consider the relation L_1 between $\mathbb{R}$ and $\mathbb{Z}$ to be: $x\ L_1\ y$ iff $x + y \leq 1$

1. Is 5 related to 7?
2. Is 1 related to 0?
3. Which x satisfy $10\ L_1\ x$?
4. Which x satisfy $x\ L_1\ 7$?

Binary Relations over Infinite Sets

Consider the relation $\boldsymbol{L_1}$ between $\mathbb{R}$ and $\mathbb{Z}$ to be: $\boldsymbol{x\ L_1\ y}$ iff $\boldsymbol{x + y \leq 1}$

1. Is 5 related to 7? **No. $5 \not{L}_1 7$, because $5 + 7 > 1$**
2. Is 1 related to 0?
3. Which $\boldsymbol{x}$ satisfy $\boldsymbol{10\ L_1\ x}$?
4. Which x satisfy $\boldsymbol{x\ L_1\ 7}$?

Binary Relations over Infinite Sets

Consider the relation L_1 between $\mathbb{R}$ and $\mathbb{Z}$ to be: $x \, L_1 \, y$ iff $x + y \leq 1$

1. Is 5 related to 7? **No.** $5 \not{L}_1 7$**, because** $5 + 7 > 1$
2. Is 1 related to 0? **Yes.** $1 \, L_1 \, 0$**, because** $1 + 0 \leq 1$
3. Which x satisfy $10 \, L_1 \, x$?
4. Which x satisfy $x \, L_1 \, 7$?

Binary Relations over Infinite Sets

Consider the relation L_1 between ℝ and ℤ to be: $x \, L_1 \, y$ iff $x + y \leq 1$

1. Is 5 related to 7? **No.** $5 \not{L_1} 7$**, because** $5 + 7 > 1$
2. Is 1 related to 0? **Yes.** $1 \, L_1 \, 0$**, because** $1 + 0 \leq 1$
3. Which x satisfy $10 \, L_1 \, x$? **All** $x \in \mathbb{Z}$ **where** $x \leq -9$
4. Which x satisfy $x \, L_1 \, 7$?

Binary Relations over Infinite Sets

Consider the relation L_1 between $\mathbb{R}$ and $\mathbb{Z}$ to be: $x\ L_1\ y$ iff $x + y \leq 1$

1. Is 5 related to 7? **No. $5 \not L_1 7$, because $5 + 7 > 1$**
2. Is 1 related to 0? **Yes. $1\ L_1\ 0$, because $1 + 0 \leq 1$**
3. Which x satisfy $10\ L_1\ x$? **All $x \in \mathbb{Z}$ where $x \leq -9$**
4. Which x satisfy $x\ L_1\ 7$? **All $x \in \mathbb{R}$ where $x \leq -6$**

Binary Relations on a Set

The binary relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is a subset of $\boldsymbol{A \times A}$.

The set $\boldsymbol{A}$ is called the **domain** of the binary relation.

Example

We can define the relation R_1 on the set of real numbers such that:

$a R_1 b$ iff $a > b$

1. Is 2 related to 3?
2. Is 5 related to 3?
3. For what values of x is x^2 related to $2x$?
4. For what values of x is x related to x?

Example

We can define the relation R_1 on the set of real numbers such that:

$a \, R_1 \, b$ iff $a > b$

1. Is 2 related to 3? **No.**
2. Is 5 related to 3? **Yes**.
3. For what values of x is x^2 related to $2x$? $x^2 > 2x$ **when** $x > 2$
4. For what values of x is x related to x? **None**

Special Properties of Binary Relations

For any binary relation, we can consider the following questions:

- Are all elements related to themselves?
- Does the relation hold in both directions?
- Does the relation only hold in one direction?
- If there is a chain of relations, does the relation also hold directly?

Special Properties of Binary Relations

For any binary relation, we can consider the following questions:

- Are all elements related to themselves? **Reflexive**
- Does the relation hold in both directions? **Symmetric**
- Does the relation only hold in one direction? **Anti-Symmetric**
- If there is a chain of relations, does the relation also hold directly? **Transitive**

Special Properties of Binary Relations

A relation ***R*** on set ***A*** is called **reflexive** if every $a \in A$ is related to itself.

Formally, $a\ R\ a$ for all $a \in A$

Example: Consider the ≤ relation on $\mathbb{Z}$

Special Properties of Binary Relations

A relation ***R*** on set ***A*** is called **symmetric** if for every ***a R b***, we also have that ***b R a***.

Example: Consider the = relation on ℤ

A relation ***R*** on set ***A*** is called **anti-symmetric** if for all ***a, b*** **∈** ***A***:
a R b and ***b R a*** implies that ***a = b***.

Example: Consider the ≤ relation on ℤ

Special Properties of Binary Relations

A relation ***R*** on set ***A*** is called **transitive** if for all ***a,b,c*** ∈ ***A***:
a R b and ***b R c*** implies ***a R c***.

Example: Consider the < relation on $\mathbb{Z}$

Exercise

Consider the following relations on the set {1,2,3}

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,1), (3,3)}

R_2 = {(1,1), (1,3), (2,2), (3,1)}

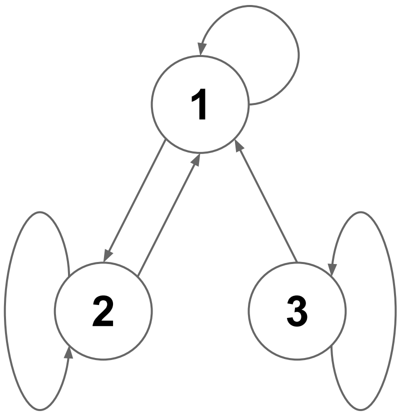
R_3 = {(2,3)}

R_4 = {(1,1), (1,3)}

What are the special properties of each relation?

Exercise

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3)\}$



Reflexive?

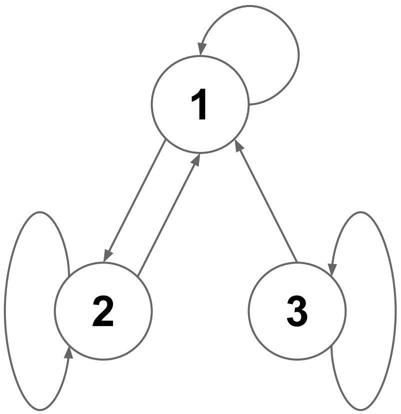
Symmetric?

Anti-Symmetric?

Transitive?

Exercise

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,1), (3,3)}



Reflexive? **Yes.**

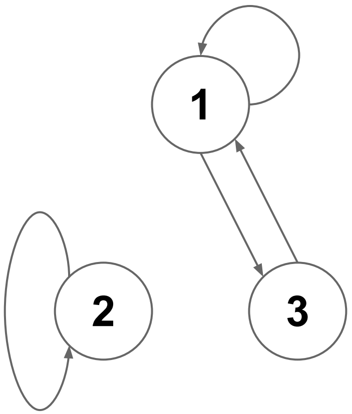
Symmetric? **No**…**3** R_1 **1** but **1** $\not R_1$ **3**

Anti-Symmetric? **No**…**2** R_1 **1** and **1** R_1 **2** but **1** $\neq$ **2**

Transitive? **No**…**3** R_1 **1** and **1** R_1 **2** but **3** $\not R_1$ **2**

Exercise

R_2 = {(1,1), (1,3), (2,2), (3,1)}



Reflexive?

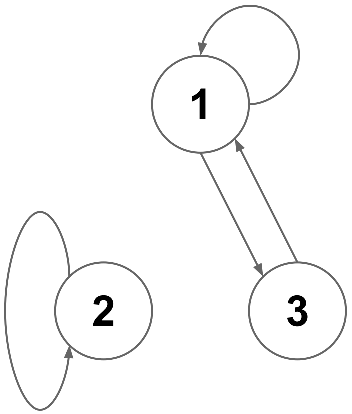
Symmetric?

Anti-Symmetric?

Transitive?

Exercise

$R_2 = \{(1,1), (1,3), (2,2), (3,1)\}$



Reflexive? **No…3** $3 \not R_2 3$

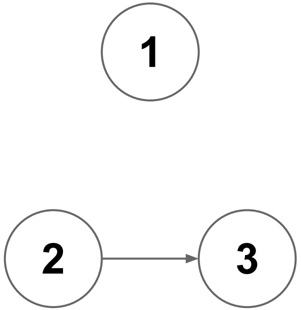
Symmetric? **Yes.** $x R_2 y \rightarrow y R_2 x$

Anti-Symmetric? **No…** $1 R_2 3$ **and** $3 R_2 1$ but $1 \neq 3$

Transitive? **No.** $3 R_2 1$ and $1 R_2 3$ but $3 \not R_2 3$

Exercise

$R_3 = \{(2,3)\}$



Reflexive?

Symmetric?

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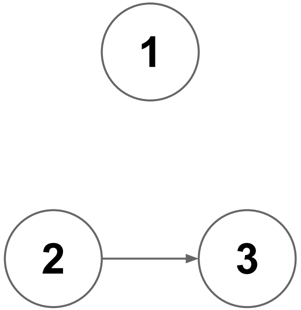
Transitive?

Exercise

$R_3 = \{(2,3)\}$



Reflexive? **No**.

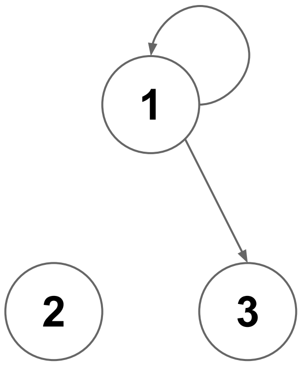
Symmetric? **No.**

Anti-Symmetric? **Yes.**

Transitive? **Yes.** Can't pick a,b,c s.t. $a\ R_3\ b$ and $b\ R_3\ c$

Exercise

$R_4 = \{(1,1), (1,3)\}$



Reflexive?

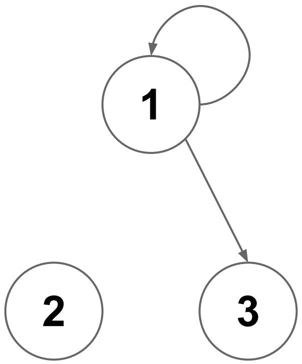
Symmetric?

Anti-Symmetric?

Transitive?

Exercise

$R_4 = \{(1,1), (1,3)\}$



Reflexive? **No**.

Symmetric? **No.**

Anti-Symmetric? **Yes.**

Transitive? **Yes. 1 R_4 1** and **1 R_4 3** $\rightarrow$ **1 R_4 3**

Outline

- **Binary Relations**
  - Intro
  - **Partial Ordering**
  - Equivalence Relations
- Functions

Partial Ordering

A relation ***R*** on a set ***A*** is called a **partial order** if it is reflexive, transitive, and antisymmetric.

a R b is denoted ***a ≤ b*** for partial a ordering ***R***

- We read ***a ≤ b*** as "***a*** is at most ***b***" or "***a*** precedes ***b***"
- A domain, ***A***, with a partial ordering **≤** can be treated as the object (***A***,**≤**)
    - (***A***,**≤**) is called a **partially ordered set** or **poset**

Partial Ordering Example

Consider the relation $\boldsymbol{R}$ on the set $\mathbb{Z}$, where:

$\boldsymbol{x}\ \boldsymbol{R}\ \boldsymbol{y}$ if and only if $\boldsymbol{x} \leq \boldsymbol{y}$

Is ($\mathbb{Z}$, $\boldsymbol{R}$) a poset?

Partial Ordering Example

Consider the relation $\boldsymbol{R}$ on the set $\mathbb{Z}$, where:

$\boldsymbol{x}\ \boldsymbol{R}\ \boldsymbol{y}$ if and only if $\boldsymbol{x} \leq \boldsymbol{y}$

Is $(\mathbb{Z}, \boldsymbol{R})$ *a poset?* **Yes.**

$\boldsymbol{R}$ is reflexive ($\boldsymbol{x} \leq \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{Z}$), $\boldsymbol{R}$ is antisymmetric ($\boldsymbol{x} \leq \boldsymbol{y}$ and $\boldsymbol{y} \leq \boldsymbol{x} \rightarrow \boldsymbol{x} = \boldsymbol{y}$) and $\boldsymbol{R}$ is transitive ($\boldsymbol{x} \leq \boldsymbol{y}$ and $\boldsymbol{y} \leq \boldsymbol{z} \rightarrow \boldsymbol{x} \leq \boldsymbol{z}$)

Comparable Elements and Total Ordering

Elements ***x*** and ***y*** are **<u>comparable</u>** if ***x*** ≤ ***y*** or ***y*** ≤ ***x*** (or both)

A partial order is a **<u>total order</u>** if every pair of elements in the domain are comparable.

In our previous example, (ℤ, ***R***) is a total order

- It is a partial order, and for every ***x,y*** ∈ ℤ, ***x R y*** or ***y R x***
- We say that ***R*** is a total ordering of ℤ

Partial vs Total Ordering

What does it look like when elements cannot be compared?

Let the operator **≤** be ⊆, where ***A* ≤ *B*** iff ***A*** ⊆ ***B***, and let ***S* = $\mathscr{P}$({a,b,c})**

We have no way to compare **{a,b}** and **{b,c}**

- **{a,b} ≰ {b,c}**
- **{b,c} ≰ {a,b}**
- Therefore, **{a,b}** and **{b,c}** are incomparable

Is (***S***, ≤) a partial ordering of ***S***?

Is (***S***, ≤) a partial ordering of ***S***?

Partial vs Total Ordering

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- **{b,c} ≰ {a,b}**
- Therefore, **{a,b}** and **{b,c}** are incomparable

Is (***S***, ≤) a partial ordering of ***S***? **Yes. ≤** is reflexive, anti-symmetric, transitive

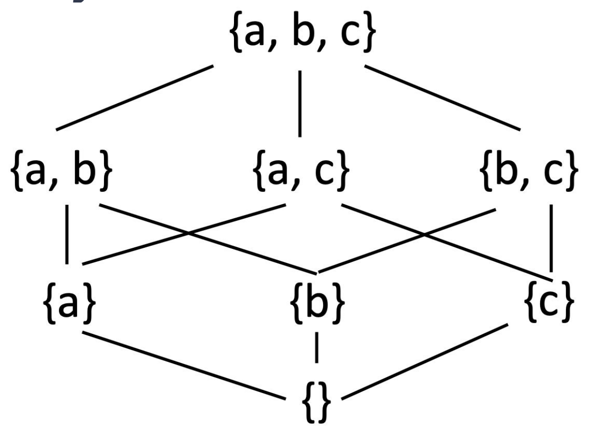
Is (***S***, ≤) a partial ordering of ***S***? **No.** There exist incomparable elements of ***S***

Hasse Diagram

Given a poset, we can draw a **Hasse Diagram** to visualize the relation

- If $x \preceq y$, then x appears lower in the drawing than y
- There is a line from x to y iff $x \preceq y$ or $y \preceq x$
- Omit line between x and z if $x \preceq z$ but $\exists y$ s.t. $x \preceq y \preceq z$

Consider $(S, \subseteq)$ where $S = \mathscr{P}(\{a,b,c\})$



Another Example

Consider the set ***H***

H = { Halloween, It, Hereditary, Get Out, Descent, Chucky, Friday the 13th }

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I could organize these movies in a tier list based on my preferences:

A-tier: Halloween, Get Out, Friday the 13th

B-tier: It, Descent, Chucky

C-tier: Hereditary

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I could organize these movies in a tier list based on my preferences:

A-tier: Halloween, Get Out, Friday the 13th

B-tier: It, Descent, Chucky

C-tier: Hereditary

Is this a partial or total ordering?
Is (Hereditary, Chucky) in the relation?
Is (Hereditary, Halloween)?
Is (Halloween, Get Out)?

Another Example

Consider the set ***H***

H = { Halloween, It, Hereditary, Get Out, Descent, Chucky, Friday the 13th }

I could organize these movies in a tier list based on my preferences:

A-tier: Halloween, Get Out, Friday the 13th

B-tier: It, Descent, Chucky

C-tier: Hereditary

Is this a partial or total ordering? partial
Is (Hereditary, Chucky) in the relation? Yes
Is (Hereditary, Halloween)? Yes (transitivity)
Is (Halloween, Get Out)? No (incomparable)

Another Example

Consider the set ***H***

H = { Halloween, It, Hereditary, Get Out, Descent, Chucky, Friday the 13th }

Another Example

Consider the set ***H***

H = { Halloween, It, Hereditary, Get Out, Descent, Chucky, Friday the 13th }

I could also rank these movies based on my preferences:

1. Halloween
2. Get Out
3. Friday the 13th
4. Descent
5. It
6. Chucky
7. Hereditary

Is this a partial or total ordering?
Is (Hereditary, Chucky) in the relation?
Is (Hereditary, Halloween)?
Is (Halloween, Get Out)?

Another Example

Consider the set ***H***

H = { Halloween, It, Hereditary, Get Out, Descent, Chucky, Friday the 13th }

I could also rank these movies based on my preferences:

1. Halloween
2. Get Out
3. Friday the 13th
4. Descent
5. It
6. Chucky
7. Hereditary

Is this a partial or total ordering? total
Is (Hereditary, Chucky) in the relation? Yes
Is (Hereditary, Halloween)? Yes (transitivity)
Is (Halloween, Get Out)? No

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  - **Equivalence Relations**
- Functions

Equivalence Relations

A relation ***R*** on a set ***A*** is called an **equivalence relation** if it is reflexive, transitive, and **symmetric**.

a R b is denoted ***a ~ b*** for an equivalence relation ***R***

- We read ***a ~ b*** as "***a*** is equivalent to ***b***"

Example

Consider the relation ***R*** on ***P* = { all people }**, where ***a R b*** iff ***a*** and ***b*** have the same birthday.

Is ***R*** an equivalence relation?

Example

Consider the relation ***R*** on ***P*** **= { all people }**, where ***a R b*** iff ***a*** and ***b*** have the same birthday.

Is ***R*** an equivalence relation? **Yes.**

Reflexive: Any person, ***a***, has the same birthday as themselves

Transitive: If person ***a*** and ***b*** have the same birthday, and ***b*** and ***c*** have the same birthday, then ***a*** and ***c*** also have the same birthday

Symmetric: If ***a R b***, then ***b R a***.

Equivalence Classes

We can partition the domain of an equivalence relation into equivalent elements. These partitions are called **equivalence classes**.

If $e \in D$ then the equivalence class containing e is denoted $[e]$

$$[e] = \{ x \mid x \in D, x \sim e \}$$

Examples

Consider the birthday equivalence relation from the previous example, ***R***

Suppose Alice's birthday is March 12

- If Alice ***R*** Bob, then Bob's birthday is also March 12
- Under the relation ***R*** Alice and Bob are equivalent (Alice ~ Bob)
- [Alice] = { Alice, Bob, … } = { all people born on March 12 }
- [Alice] = [Bob] since both represent people born on March 12

Examples

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Do the equivalence classes form a partition of the domain?

Examples

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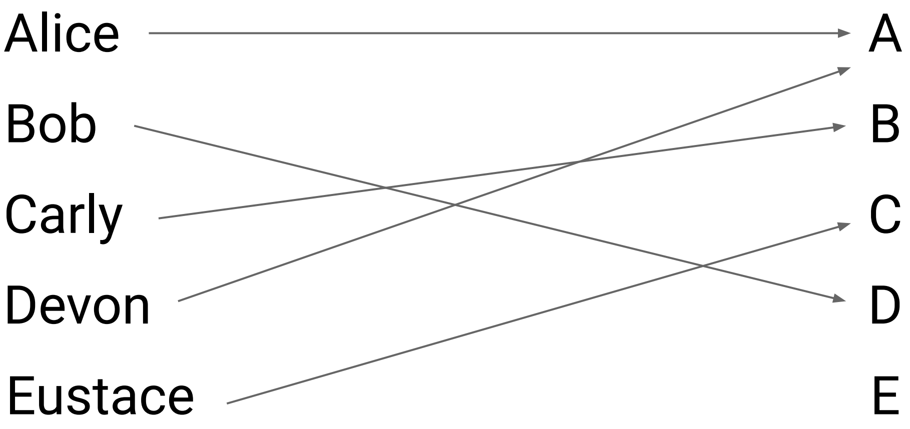
Do the equivalence classes form a partition of the domain? **Yes**

Outline

- Binary Relations
- **Functions**
  - **Introduction to Functions**
  - Function Equality
  - Function Properties
  - Floor/Ceiling Functions
  - Division of Modular Arithmetic
  - Composition of Functions

Function Definition

Consider a relation of students to letter grades:



We may want to be able to input a student's name a get their grade
(A function is a different take on binary relations)

Function Definition

Let ***A*** and ***B*** be nonempty sets. A **function**, f, from ***A*** to ***B*** is an assignment of exactly one element of ***B*** to each element of ***A***.

Denoted by f: ***A*** $\longrightarrow$ ***B***

We write f(***a***) = ***b*** if ***b*** is the **unique element** of ***B*** assigned by f to the element ***a*** of ***A***

The set ***A*** is the **domain** of f

The set ***B*** is the **codomain** of f

Function Examples

Consider the sets $\boldsymbol{X_1} = \{1, 2, 3\}$, $\boldsymbol{Y_1} = \{1, 2, 3\}$, and the mapping $f_1: \boldsymbol{X_1} \to \boldsymbol{Y_1}$:

$$f_1(\boldsymbol{x}) = \boldsymbol{x}$$

1. Is f_1 a function?
2. What is the domain of f_1?
3. What is the codomain of f_1?

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2. What is the domain of f_1? $\boldsymbol{X_1}$
3. What is the codomain of f_1? $\boldsymbol{Y_1}$

We can also write $f_1 = \{(1,1), (2,2), (3,3)\}$

Function Examples

Consider the sets $\boldsymbol{X_2} = \mathbb{Z}$, $\boldsymbol{Y_2} = \{1, 2, 3\}$, and the mapping f_2: $\boldsymbol{X_2} \to \boldsymbol{Y_2}$:

$$f_2(\boldsymbol{x}) = \boldsymbol{x}$$

1. Is f_2 a function?

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We could fix this by defining a new mapping:

$$f_2'(x) = \begin{cases} x & \text{if } 1 \leq x \leq 3 \\ 1 & \text{if } x < 1 \text{ or } x > 3 \end{cases}$$

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$f_2' = \{(1,1), (2,2), (3,3)\} \cup \{(\boldsymbol{x}, 1) \mid \boldsymbol{x} \in \mathbb{Z}, \boldsymbol{x} < 1 \text{ or } \boldsymbol{x} > 3\}$

Function Examples

Consider the sets $\boldsymbol{X_3} = \mathbb{Z}$, $\boldsymbol{Y_3} = \mathbb{Z}$, and the mapping f_3: $\boldsymbol{X_3} \rightarrow \boldsymbol{Y_3}$:

$$f_3(x) = \begin{cases} x & \text{if } x \text{ is odd} \\ x^2 & \text{if } x \geq 0 \\ |x| & \text{if } x < 0 \end{cases}$$

Is f_3 a function?

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Is f_3 a function? **No.** Problem: $f_3(-1) = -1$ and $f_3(-1) = 1$

Function Definition in Symbols

Symbolically, for a mapping f: $\boldsymbol{X} \to \boldsymbol{Y}$:

f is a (well-defined) function if and only if

$$\forall \boldsymbol{x} \in \boldsymbol{X},\ \exists \boldsymbol{y} \in \boldsymbol{Y},\ (f(x) = y \land (\forall \boldsymbol{z} \in \boldsymbol{Y},\ (\boldsymbol{y} \neq \boldsymbol{z} \to f(x) \neq \boldsymbol{z})))$$

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For every ***x*** in the domain,
there exists a ***y*** in the
codomain such that $f(x) = y$

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f is a (well-defined) function if and only if

$$\forall \boldsymbol{x} \in \boldsymbol{X},\ \exists \boldsymbol{y} \in \boldsymbol{Y},\ (f(\mathrm{x}) = \mathrm{y} \wedge (\forall \boldsymbol{z} \in \boldsymbol{Y},\ (\boldsymbol{y} \neq \boldsymbol{z} \rightarrow f(\mathrm{x}) \neq \boldsymbol{z})))$$

For every ***x*** in the domain, there exists a ***y*** in the codomain such that f(x) = y

For every other element of the codomain, ***z***, f(x) ≠ ***z***.

Function Range

If f is a function from $\boldsymbol{A}$ to $\boldsymbol{B}$, the set range(f) = $\{ \boldsymbol{y} \mid \exists \boldsymbol{x} \in \boldsymbol{A}, f(x) = y \}$ is called the **range** of f

It is the set of all values in the codomain that have an element from the domain mapped to it

- For any function f: $\boldsymbol{A} \rightarrow \boldsymbol{B}$, range($f$) $\subseteq \boldsymbol{B}$
- It does not have to be the whole codomain

More Examples

$\boldsymbol{X_4} = \mathbb{Z}$, $\boldsymbol{Y_4} = \mathbb{Z}$

$f_4(x)$: $\boldsymbol{X_4} \rightarrow \boldsymbol{Y_4}$

$f_4(x) = 1$

Is f_4 a function?

Domain?

Codomain?

Range?

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Is f_4 a function? **Yes**

Domain? $\mathbb{Z}$

Codomain? $\mathbb{Z}$

Range?

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$\boldsymbol{X_4} = \mathbb{Z}$, $\boldsymbol{Y_4} = \mathbb{Z}$

$f_4(x)$: $\boldsymbol{X_4} \rightarrow \boldsymbol{Y_4}$

$f_4(x) = 1$

Is f_4 a function? **Yes**

Domain? $\mathbb{Z}$

Codomain? $\mathbb{Z}$

Range? $\{1\}$

More Examples

$\boldsymbol{X_5} = \mathbb{Z}$, $\boldsymbol{Y_5} = \mathbb{Z}$

$f_5(x)$: $\boldsymbol{X_5} \rightarrow \boldsymbol{Y_5}$

$f_5(x) = \sqrt{x}$

Is f_5 a function?

Domain?

Codomain?

Range?

More Examples

$\boldsymbol{X_5} = \mathbb{Z}$, $\boldsymbol{Y_5} = \mathbb{Z}$

$f_5(x)$: $\boldsymbol{X_5} \rightarrow \boldsymbol{Y_5}$

$f_5(x) = \sqrt{x}$

Is f_5 a function? **No**

Domain? **N/A**

Codomain? **N/A**

Range? **N/A**

More Examples

$\boldsymbol{X_6} = \{ \boldsymbol{x}^2 \mid \boldsymbol{x} \in \mathbb{Z} \}$

$\boldsymbol{Y_6} = \mathbb{Z}$

$f_6(x)\colon \boldsymbol{X_6} \to \boldsymbol{Y_6}$

$f_6(x) = \sqrt{x}$

Is f_6 a function?

Domain?

Codomain?

Range?

More Examples

$\boldsymbol{X_6} = \{ \boldsymbol{x^2} \mid \boldsymbol{x} \in \mathbb{Z} \}$

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$f_6(x)\colon \boldsymbol{X_6} \rightarrow \boldsymbol{Y_6}$

$f_6(x) = \sqrt{x}$

Is f_6 a function? **Yes**

Domain?

Codomain?

Range?

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Is f_6 a function? **Yes**

Domain? $\{ \boldsymbol{x}^2 \mid \boldsymbol{x} \in \mathbb{Z} \}$

Codomain? $\mathbb{Z}$

Range?

More Examples

$\boldsymbol{X_6} = \{\, \boldsymbol{x^2} \mid \boldsymbol{x} \in \mathbb{Z} \,\}$

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$f_6(x)\colon \boldsymbol{X_6} \to \boldsymbol{Y_6}$

$f_6(x) = \sqrt{x}$

Is f_6 a function? **Yes**

Domain?	$\{\, \boldsymbol{x^2} \mid \boldsymbol{x} \in \mathbb{Z} \,\}$
Codomain?	$\mathbb{Z}$
Range?	$\mathbb{Z}^+ \cup \{0\}$

Outline

- Binary Relations
- **Functions**
    - Introduction to Functions
    - **Function Equality**
    - Function Properties
    - Floor/Ceiling Functions
    - Division of Modular Arithmetic
    - Composition of Functions

Function Equality

Two functions, $f\colon \boldsymbol{A} \to \boldsymbol{B}$ and $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ are equal iff the following hold:

1. $\boldsymbol{A} = \boldsymbol{X}$
2. $\boldsymbol{B} = \boldsymbol{Y}$
3. $\forall \boldsymbol{a} \in \boldsymbol{A}, f(\boldsymbol{a}) = g(\boldsymbol{a})$

In English, two functions are equal if they have the same domain, same codomain, and map each element in the domain to the same element of the codomain

Function Equality

Consider the functions: $f\colon \mathbb{Z} \to \mathbb{Z}$, and $g\colon \mathbb{Z} \to \mathbb{Z}$ defined as:

$$f = \{(x, 1) \mid x \in \mathbb{Z}\} \quad \text{and} \quad g(y) = 1$$

Are the two functions equal?

Function Equality

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Are the two functions equal?

1. Same domain ✓

Function Equality

Consider the functions: f: $\mathbb{Z} \to \mathbb{Z}$, and g: $\mathbb{Z} \to \mathbb{Z}$ defined as:

$$f = \{(x, 1) \mid x \in \mathbb{Z}\} \quad \text{and} \quad g(y) = 1$$

Are the two functions equal?

1. Same domain ✓
2. Same codomain ✓
3. $\forall \boldsymbol{x} \in \mathbb{Z}, f(\boldsymbol{x}) = g(\boldsymbol{x})$?

Function Equality

Consider the functions: $f: \mathbb{Z} \rightarrow \mathbb{Z}$, and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ defined as:

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Are the two functions equal?

1. Same domain ✓
2. Same codomain ✓
3. $\forall \boldsymbol{x} \in \mathbb{Z}, f(\boldsymbol{x}) = g(\boldsymbol{x})$?

- Pick an arbitrary $\boldsymbol{x} \in \mathbb{Z}$
- Then $(\boldsymbol{x}, 1) \in f$, or $f(\boldsymbol{x}) = 1$
- Similarly $g(\boldsymbol{x}) = 1$

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Injective Functions

A function f: $A \rightarrow B$ is **injective** if $\forall x_1, x_2 \in A, (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$

Also known as **one-to-one** or **1-1**

- Each element in the domain is mapped to a unique element from the codomain (no element in the codomain is hit twice)
- To prove a function is 1-1
  - Take an arbitrary x and y such that $f(x) = f(y)$
  - Conclude that $x = y$
- To prove a function is not 1-1
  - Find a counterexample where $x \neq y$ but $f(x) = f(y)$

Surjective Functions

A function f: $A \to B$ is **surjective** if $\forall y \in B,\ \exists x \in A, f(x) = y$

Also known as **onto**

- Every element in the codomain has an element that maps to it
- To prove a function is onto:
  - Take arbitrary y in the codomain
  - Find the value of x in the domain such that $f(x) = y$
- To prove a function is not onto:
  - Find a counterexample, element y in codomain s.t. no element maps to it

Bijective Functions

Idea: What if a function is both 1-1 and onto?

Bijective Functions

A function $f: \boldsymbol{A} \rightarrow \boldsymbol{B}$ is **bijective** if it is injective and surjective

A bijective function is called a **bijection**, or a **one-to-one correspondence**

Examples

Let $f_1 \colon \mathbb{Z} \to \mathbb{Z}$, defined by $f_1(a) = 1$

Is f_1 injective, surjective, bijective?

Injective (1-1)?

Examples

Let $f_1 : \mathbb{Z} \rightarrow \mathbb{Z}$, defined by $f_1(a) = 1$

Is f_1 injective, surjective, bijective?

Injective (1-1)?

1. Let x = 1 and y = 5 (clearly x and y are both in $\mathbb{Z}$)
2. $f_1(x) = 1, f_1(y) = 1$
3. Therefore $f_1(x) = f_1(y)$, but $x \neq y$

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We've found a counterexample, so f_1 is not 1-1

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Surjective (onto)?

1. Consider y = 2 (clearly 2 is in the codomain $\mathbb{Z}$)
2. There is no $x \in \mathbb{Z}$ s.t. $f_1(x) = y$

Examples

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Bijective (one-to-one correspondence)?

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Is f_1 injective, surjective, bijective?

Bijective (one-to-one correspondence)?

No. To be bijective f_1 must be injective AND surjective

Examples

Let f_2: $\mathbb{Z} \rightarrow \mathbb{Z}^+$, defined by f_2(a) = |a| (absolute value of a)

Is f_2 injective, surjective, bijective?

Injective (1-1)?

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Is f_2 injective, surjective, bijective?

Injective (1-1)?

1. Let x = 2 and y = -2 (clearly x and y are both in $\mathbb{Z}$)
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3. Therefore $f_1(x) = f_1(y)$, but $x \neq y$

We've found a counterexample, so f_2 is not 1-1

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Is f_2 injective, surjective, bijective?

Surjective (onto)?

1. Let y be an arbitrary element of $\mathbb{Z}^+$
2. f_2(y) = y

Examples

Let f_2: $\mathbb{Z} \rightarrow \mathbb{Z}^+$, defined by f_2(a) = |a| (absolute value of a)

Is f_2 injective, surjective, bijective?

Surjective (onto)?

1. Let y be an arbitrary element of $\mathbb{Z}^+$
2. $f_2(y) = y$

Therefore, since we chose y arbitrarily, every element of $\mathbb{Z}^+$ gets mapped to by something, therefore f_2 is onto

Examples

Let f_2: $\mathbb{Z} \rightarrow \mathbb{Z}^+$, defined by f_2(a) = |a| (absolute value of a)

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Bijective (one-to-one correspondence)?

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Examples

Let f_3: $\mathbb{Z} \rightarrow \mathbb{Z}$, defined by f_3(a) = a + 16

Is f_3 injective, surjective, bijective?

Injective (1-1)?

Examples

Let f_3: $\mathbb{Z} \rightarrow \mathbb{Z}$, defined by $f_3(a) = a + 16$

Is f_3 injective, surjective, bijective?

Injective (1-1)?

1. Let x and y be arbitrary elements of $\mathbb{Z}$
2. Assume $f_3(x) = f_3(y)$
3. $x + 16 = y + 16 \rightarrow x = y$

Examples

Let f_3: $\mathbb{Z} \rightarrow \mathbb{Z}$, defined by $f_3(a) = a + 16$

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Injective (1-1)?

1. Let x and y be arbitrary elements of $\mathbb{Z}$
2. Assume $f_3(x) = f_3(y)$
3. $x + 16 = y + 16 \rightarrow x = y$

Therefore, since we chose x and y arbitrarily, and $f_3(x) = f_3(y) \rightarrow x = y$, then f_3 is 1-1

Examples

Let f_3: $\mathbb{Z} \longrightarrow \mathbb{Z}$, defined by f_3(a) = a + 16

Is f_3 injective, surjective, bijective?

Surjective (onto)?

Examples

Let f_3: $\mathbb{Z} \rightarrow \mathbb{Z}$, defined by $f_3(a) = a + 16$

Is f_3 injective, surjective, bijective?

Surjective (onto)?

1. Let y be an arbitrary element of $\mathbb{Z}$
2. $x = y - 16$ is also therefore in $\mathbb{Z}$
3. $f_3(x) = x + 16 = (y - 16) + 16 = y$

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Is f_3 injective, surjective, bijective?

Bijective (one-to-one correspondence)?

Yes. To be bijective f_3 must be injective AND surjective, and it is!

Inverse of Functions

For any function f: $\boldsymbol{A} \rightarrow \boldsymbol{B}$, the **inverse mapping** of f, denoted by f^{-1}, is defined by the mapping f^{-1}: $\boldsymbol{B} \rightarrow \boldsymbol{A}$ where: $f^{-1} = \{ (\boldsymbol{y}, \boldsymbol{x}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in f \}$

If f is a **bijection** then f^{-1} is a function (otherwise it is just a mapping)

- f^{-1} maps codomain elements of f to domain elements of f
- If $f(x) = y$ then $f^{-1}(y) = x$

Inverse of Functions

For f^{-1} to be a function, f must be a bijection:

- 1-1: guarantees at most one arrow out of each codomain element
- Onto: guarantees at least one arrow out of each codomain element
- Bijection: **exactly** one arrow out of each codomain element

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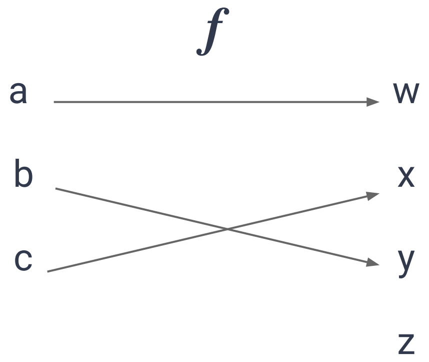

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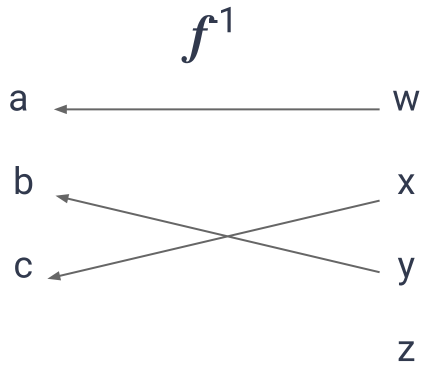

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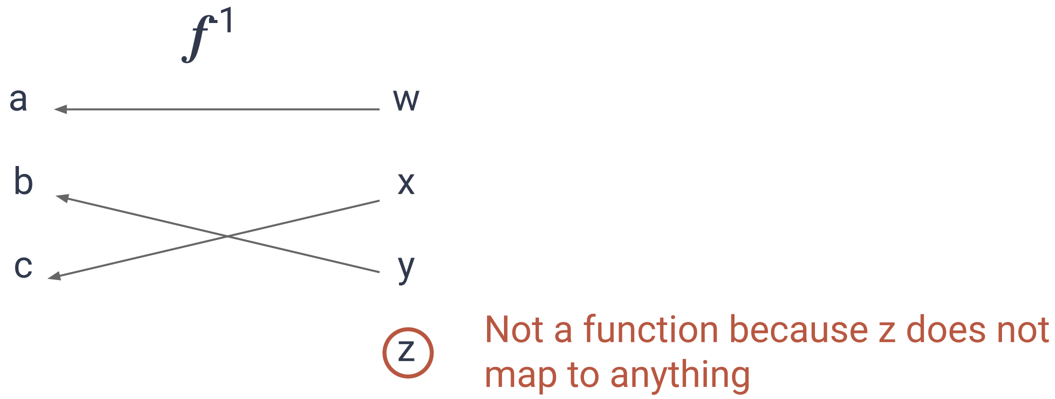

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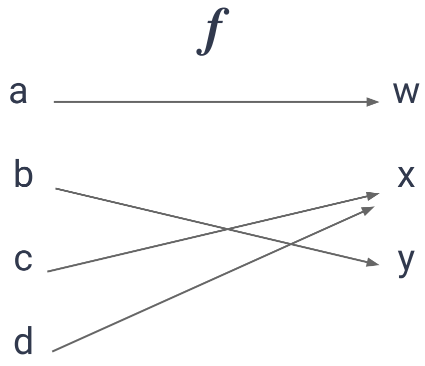

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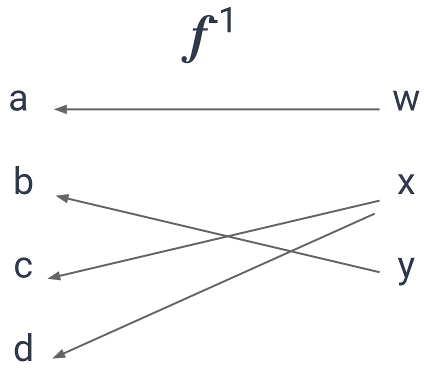

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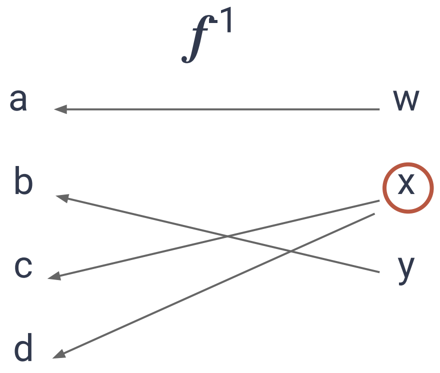

Not a function because x maps to more than one thing

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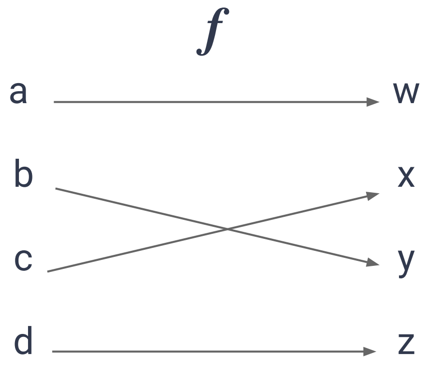

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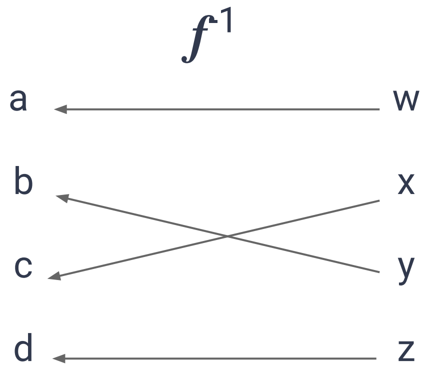

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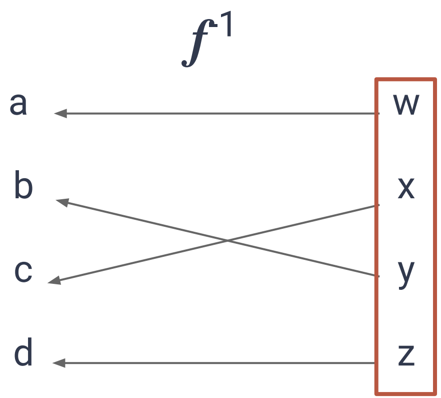

Exactly one arrow out of every element of codomain, therefore f^{-1} is a function

Cardinality of Domain vs Codomain

If f: $\boldsymbol{A} \rightarrow \boldsymbol{B}$ is **onto:**

- Then for every codomain element, there is at least one domain element
- $|\boldsymbol{A}| \geq |\boldsymbol{B}|$

If f: $\boldsymbol{A} \rightarrow \boldsymbol{B}$ is **1-1**:

- For every domain element, there is a unique codomain element
- $|\boldsymbol{A}| \leq |\boldsymbol{B}|$

If f: $\boldsymbol{A} \rightarrow \boldsymbol{B}$ is a **bijection**, then f is **1-1** and **onto**

- $|\boldsymbol{A}| \leq |\boldsymbol{B}|$ and $|\boldsymbol{A}| \geq |\boldsymbol{B}|$, therefore $|\boldsymbol{A}| = |\boldsymbol{B}|$

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This will be useful for comparing the cardinality of sets!

Cardinality of $\mathbb{N}$ and $\mathbb{Z}$

Theorem: The cardinalities of $\mathbb{N}$ and $\mathbb{Z}$ are the same

Proof: Show a bijection from $\mathbb{N}$ to $\mathbb{Z}$

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Let f: $\mathbb{N} \longrightarrow \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} -i & \text{if } x = 2i \text{ (x is even)} \\ i & \text{if } x = 2i + 1 \text{ (x is odd)} \end{cases}$$

Is f a bijection?

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$$f(x) = \begin{cases} -i & \text{if } x = 2i \text{ (x is even)} \\ i & \text{if } x = 2i + 1 \text{ (x is odd)} \end{cases}$$

Let ***x*** and ***y*** be arbitrary elements of $\mathbb{N}$

Assume *f*(***x***) = *f*(***y***). Then ***x*** and ***y*** must both be even, or both be odd.

If even: ***x*** = 2***i*** and ***y*** = 2***i***.

If odd: ***x*** = 2***i*** + 1 and ***y*** = 2***i*** + 1.

Therefore, if *f*(***x***) = *f*(***y***), ***x*** = ***y***, which means *f* is 1-1

Cardinality of $\mathbb{N}$ and $\mathbb{Z}$

$$f(x) = \begin{cases} -i & \text{if } x = 2i \text{ (x is even)} \\ i & \text{if } x = 2i + 1 \text{ (x is odd)} \end{cases}$$

Let ***z*** be an arbitrary element of $\mathbb{Z}$

Case 1: ***z*** < 0

x = 2 * **-*z*** is an integer > 0, therefore ***x*** is in $\mathbb{N}$ and *f*(***x***) = ***z***

Case 2: ***z*** ≥ 0

x = 2 * ***z*** + 1 is an integer > 0, therefore ***x*** is in $\mathbb{N}$ and *f*(***x***) = ***z***

Therefore for any arbitrary ***z*** in $\mathbb{Z}$, exists an ***x*** in $\mathbb{N}$ s.t. *f*(***x***) = ***z***. So *f* is onto.

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Therefore f is both 1-1 and onto. So f is a bijection.
Therefore $\mathbb{N}$ and $\mathbb{Z}$ have the same size.

Outline

- Binary Relations
- **Functions**
  - Introduction to Functions
  - Function Equality
  - Function Properties
  - **Floor/Ceiling Functions**
  - Division of Modular Arithmetic
  - Composition of Functions

Floor Functions

The floor function is the function floor : $\mathbb{R} \rightarrow \mathbb{Z}$ defined by

$$\text{floor}(\boldsymbol{x}) = \max\{\, \boldsymbol{y} \mid \boldsymbol{y} \in \mathbb{Z}, \boldsymbol{y} \leq \boldsymbol{x} \,\}$$

Evaluates to the maximum integer below the given number.

Denoted by: floor($\boldsymbol{x}$) = $\lfloor \boldsymbol{x} \rfloor$

Examples

$\lfloor 4.5 \rfloor = 4$

$\lfloor 17 \rfloor = 17$

$\lfloor -8.7 \rfloor = -9$

$\lfloor \boldsymbol{\pi} \rfloor = \lfloor 3.14159 \rfloor = 3$

Ceiling Function

The ceiling function is the function floor : $\mathbb{R} \to \mathbb{Z}$ defined by

$$\text{ceiling}(\boldsymbol{x}) = \min\{\, \boldsymbol{y} \mid \boldsymbol{y} \in \mathbb{Z}, \boldsymbol{y} \geq \boldsymbol{x} \,\}$$

Evaluates to the minimum integer above the given number.

Denoted by: ceiling($\boldsymbol{x}$) = $\lceil \boldsymbol{x} \rceil$

Examples

$\lceil 4.5 \rceil = 5$

$\lceil 17 \rceil = 17$

$\lceil -8.7 \rceil = -8$

$\lceil \boldsymbol{\pi} \rceil = \lceil 3.141592 \rceil = 4$

Outline

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Divides

Let x and y be integers. Then x **divides** y if there is an integer k s.t. $y = kx$.

Denoted by $x \mid y$

- x does not divide y is denoted by $x \nmid y$

If $x \mid y$, then we say:

- y is a multiple of x
- x is a factor or divisor of y

Divides Examples

The question “$x \mid y$?” asks “Does x divide y?”

Examples

- $4 \mid 8$? **Yes!** $8 = 2 \cdot 4$
- $5 \mid 15$? **Yes!** $15 = 3 \cdot 5$
- $6 \mid 15$? **No!** $15 = 2 \cdot 6 + 3$

Division Algorithm

Theorem

Let $n \in \mathbb{Z}$ and let $d \in \mathbb{Z}^+$

Then there are unique integers q and r, with $0 \le r \le d - 1$, s.t. $n = q \cdot d + r$.

- If $x \mid y$ then $r = 0$, i.e., $y = qx + 0$ for some $q \in \mathbb{Z}$
- If $x \nmid y$ then $r \neq 0$, i.e., $y = qx + r$ for some $q \in \mathbb{Z}$ and $1 \le r \le x - 1$.

Even vs odd: $2 \mid x$?

$$x = \begin{cases} 2q + 0 & \text{if } x \text{ is even} \\ 2q + 1 & \text{if } x \text{ is odd} \end{cases}$$

Integer Division Definition

The Division Algorithm for $\boldsymbol{n} \in \mathbb{Z}$ and $\boldsymbol{d} \in \mathbb{Z}^+$ gives unique values

$$\boldsymbol{q} \in \mathbb{Z} \text{ and } \boldsymbol{r} \in \{0, \dots, \boldsymbol{d} - 1\}.$$

- The number $\boldsymbol{q}$ is called the quotient.
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- The number ***q*** is called the quotient.
- The number ***r*** is called the remainder.

The operations ***div*** and ***mod*** produce the quotient and the remainder, respectively, as a function of ***n*** and ***d***.

- ***n div d = q***
- ***n mod d = r***

In programming, ***n*** % ***d*** = ***r*** denotes ***n mod d = r***

Addition mod *n*

For any integer $n > 0$, ***x mod n*** can be seen as a function $\boldsymbol{mod_n(x)}$:

- $\boldsymbol{mod_n}: \mathbb{Z} \to \{0, 1, 2, \ldots, n - 1\}$, where $\boldsymbol{mod_n(x) = x \text{ mod } n}$.

<u>**Addition**</u> ***mod n*** is defined by adding two numbers and then applying $\boldsymbol{mod_n}$

- All results in the range $\{0, 1, \ldots, n - 1\}$

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Suppose $n = 7$

$+mod_7(4, 6) =$

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$+mod_7(8, 20) = (8 + 20) \bmod 7 = 28 \bmod 7 = 0$

Multiplication mod *n*

Multiplication ***mod n*** is defined by multiplying two numbers and then applying $\boldsymbol{mod}_n$.

- All results in the range { 0, 1, ... , ***n*** − 1 }

Suppose ***n*** = 11.

∗ $\boldsymbol{mod}_{11}(4, 6) =$

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∗ $\boldsymbol{mod}_{11}(8, 23) =$

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$* \; \textbf{\textit{mod}}_{11}(8, 23) = (8 * 23) \; \textbf{\textit{mod}} \; 11 = 184 \; \textbf{\textit{mod}} \; 11 = 8$

Congruence Modulo

If ***a*** and ***b*** are integers and ***m*** is a positive integer, then ***a* is congruent to *b* modulo *m*** if ***m*** divides ***a*** − ***b***.

The notation ***a*** ≡ ***b*** (***mod m***) indicates that ***a*** is congruent to ***b*** modulo ***m***.

- ***a*** ≡ ***b*** (***mod m***) is a congruence.
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Is 17 congruent to 5 modulo 6?

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Is 17 congruent to 5 modulo 6? **Yes**, because 6 divides 17 - 5.

- 17 ***mod*** 6 = 5; it is in the equivalence class for 5 in ***mod*** 6.
- 5 ***mod*** 6 = 5; it is in the equivalence class for 5 in ***mod*** 6.

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Composition of Functions

If f and g are two functions, where f: $X \rightarrow Y$ and g: $Y \rightarrow Z$, the **composition of g with f,** denoted by $g \circ f$, is the function:

$$(g \circ f): X \rightarrow Z, \text{ s.t. for all } x \in X, (g \circ f)(x) = g(f(x))$$

Example

Consider the functions f: $\mathbb{Z}^+ \rightarrow \mathbb{Z}^-$ and g: $\mathbb{Z} \rightarrow$ **{ 0, 1}** where:

$$f(x) = -x$$

$$g(x) = \begin{cases} 0 & \text{if } \lfloor \frac{x}{2} \rfloor = \frac{x}{2} \\ 1 & \text{otherwise} \end{cases}$$

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Then we can work out that:

$$(g \circ f)(x) = g(f(x)) = g(-x) = \begin{cases} 0 & \text{if } \left\lfloor \frac{-x}{2} \right\rfloor = \frac{-x}{2} \\ 1 & \text{otherwise} \end{cases}$$

Interesting Function Composition

Idea: Compose the inverse function (if it exists) with the original function

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A function mapping an element to itself is called an **identity function**

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Is $(\boldsymbol{f} \circ \boldsymbol{f}^{-1})$ also an identity function?

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Is $(f \circ f^{-1})$ also an identity function? **Yes.** $\mathbf{I}_{Y}$