CSE 191
Introduction to Discrete Structures

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Sequences, Summations, and Induction
Mid-Semester Review

Topics Covered so Far...

1. Propositional and Predicate Logic
2. Logical and Mathematical Proofs
3. Sets
4. Functions and Relations
Outline

- **Sequences**
  - **Definition and Terminology**
  - Finite Sequences
  - Infinite Sequences
  - Explicit Formulas
- Summations
- Recurrence Relations
- Mathematical Induction
- Strong Induction
Sequences

Example: Let's say you want to represent your GPA over four semester...

Consider the function \( g: \{1,2,3,4\} \rightarrow \{ \text{gpa} \mid \text{gpa} \in \mathbb{R}, 0 \leq \text{gpa} \leq 4 \} \) defined by:

\[
g(1) = 3.6 \quad g(2) = 2.8 \quad g(3) = 3.2 \quad g(4) = 3.8
\]

For shorthand, we write \( a(n) \) as \( a_n \)

So \( g_1, g_2, g_3, g_4 \) represents our GPA sequence, 3.6, 2.8, 3.2, 3.8
A **sequence** is created by a special type of function with a domain of consecutive integers...ie no gaps in the domain

**OK:** \( \mathbb{N}, \mathbb{Z}^+, \mathbb{Z}^+ \cup \{0\}, \mathbb{Z}, \{1,2,3,4,5\} \)

**Not OK:** \( \{1,3,5,7\}, \{x \in \mathbb{N} \mid x \text{ is even}\} \)
Sequences: Terminology

Given a sequence, \( a \) over domain \( D \subseteq \mathbb{Z} \)

- A **single term** (or **term**) \( a(k) \) is written as \( a_k \) for any \( k \in D \)
- Given term \( a_k \), \( k \) denotes the **index** of \( a_k \)
- A shorthand for the entire sequence is \( \{ a_k \} \) or \( \{ a_n \} \)
Sequences: Ground Rules

- Most sequences start at index 0 or 1
  - The domain is typically $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$
  - In this case we write $a_1, a_2, a_3, \ldots$ or $b_0, b_1, b_2, \ldots$

- Sequences can start from any integer
  - A sequence starting at -2: $a_{-2}, a_{-1}, a_0, a_1, \ldots$
  - A sequence starting at 5: $b_5, b_6, b_7, b_8, \ldots$

- The term $b_8$ might not necessarily be the 8th term
  - In $b_5, b_6, b_7, b_8, \ldots$ the term $b_8$ is the 4th term
Outline

- **Sequences**
  - Definition and Terminology

- **Finite Sequences**
  - Infinite Sequences
  - Explicit Formulas

- Summations
- Recurrence Relations
- Mathematical Induction
- Strong Induction
Sequences: Finite

A sequence with a finite domain is called a **finite sequence**.

Consider \( a_m, a_{m+1}, a_{m+2}, \ldots, a_n \)

There is an **initial index** \( m \) and \( a_m \) denotes the **initial term**

There is a **final index** \( n \) and \( a_n \) denotes the **final term**

**Examples:**
GPA over 4 semesters
1,2,3,4
5,4,5,4,5,4,5
Outline

- **Sequences**
  - Definition and Terminology
  - Finite Sequences
  - **Infinite Sequences**
    - Explicit Formulas
- Summations
- Recurrence Relations
- Mathematical Induction
- Strong Induction
A sequence with an infinite domain is called an infinite sequence.

Consider \( \ldots, a_m, a_{m+1}, a_{m+2}, \ldots \)

There may or may not be an initial index \( m \) (and initial term \( a_m \))

There may or may not be an final index \( n \) (and final term \( a_n \))

Examples:

1, 3, 5, 7, ... (positive odd numbers)
2, 4, 6, 8, ... (positive even numbers)
\{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, ... (the sets \( A_i \) we had previously defined)
Outline

- **Sequences**
  - Definition and Terminology
  - Finite Sequences
  - Infinite Sequences
  - **Explicit Formulas**
- Summations
- Recurrence Relations
- Mathematical Induction
- Strong Induction
Sequences: Explicit

An **explicit formula** lets us compute the value of term \( a_k \) as a function of \( k \).

**Examples:**

\( c_k = 5 \) for \( 1 \leq k \leq 5 \) \quad \rightarrow \{ c_k \} = 5,5,5,5,5

\( d_k = k \) for \( 1 \leq k \leq 10 \) \quad \rightarrow \{ d_k \} = 1,2,3,4,5,6,7,8,9,10

\( e_k = 2k \) for \( k \geq 1 \) \quad \rightarrow \{ e_k \} = 2,4,6,8, ... 

\( f_k = 2^k \) for \( k \geq 0 \) \quad \rightarrow \{ f_k \} = 1,2,4,8,16, ...
Increasing Sequences

A sequence \( \{a_k\} \) is **increasing** if, \( \forall i, a_i < a_{i+1} \)

A sequence \( \{a_k\} \) is **non-decreasing** if, \( \forall i, a_i \leq a_{i+1} \)
Increasing Sequences

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A sequence \( \{a_k\} \) is **non-decreasing** if, \( \forall i, a_i \leq a_{i+1} \)

d_k = k \text{ for } 1 \leq k \leq 10 \text{ is increasing}

e_k = 2k \text{ for } k \geq 1 \text{ is increasing}

f_k = 2^k \text{ for } k \geq 0 \text{ is increasing}
Increasing Sequences

A sequence $\{a_k\}$ is \textbf{increasing} if, $\forall i, a_i < a_{i+1}$

A sequence $\{a_k\}$ is \textbf{non-decreasing} if, $\forall i, a_i \leq a_{i+1}$

$d_k = k$ for $1 \leq k \leq 10$ is \textbf{increasing}

$e_k = 2k$ for $k \geq 1$ is \textbf{increasing}

$f_k = 2^k$ for $k \geq 0$ is \textbf{increasing}

What about the sequence $\{h_k\} = 1, 2, 2, 2, 3$?
Increasing Sequences

A sequence \{a_k\} is **increasing** if, \( \forall i, a_i < a_{i+1} \)

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What about the sequence \{ h_k \} = 1, 2, 2, 2, 3? **Non-Decreasing**
Increasing Sequences

A sequence \( \{a_k\} \) is **increasing** if, \( \forall i, a_i < a_{i+1} \)

A sequence \( \{a_k\} \) is **non-decreasing** if, \( \forall i, a_i \leq a_{i+1} \)

\( d_k = k \) for \( 1 \leq k \leq 10 \) is increasing

\( e_k = 2k \) for \( k \geq 1 \) is increasing

\( f_k = 2^k \) for \( k \geq 0 \) is increasing

What about the sequence \( \{h_k\} = 1, 2, 2, 2, 3 \)? **Non-Decreasing**

Every increasing sequence is non-decreasing

Not every non-decreasing sequence is increasing
A sequence \( \{a_k\} \) is **decreasing** if, \( \forall i, a_i > a_{i+1} \)

A sequence \( \{a_k\} \) is **non-increasing** if, \( \forall i, a_i \geq a_{i+1} \)
A sequence \( \{a_k\} \) is **decreasing** if, \( \forall i, a_i > a_{i+1} \)

A sequence \( \{a_k\} \) is **non-increasing** if, \( \forall i, a_i \geq a_{i+1} \)

- \( s_k = 10 - k \) for \( 1 \leq k \leq 10 \) is decreasing (and non-increasing)
- \( t_k = -2k \) for \( k \geq 1 \) is decreasing (and non-increasing)
- \( u_k = 2^{-k} \) for \( k \geq 0 \) is decreasing (and non-increasing)

What about the sequence \( \{vh_k\} = 3, 2, 2, 2, 1 \)? Non-Increasing
A geometric sequence is a sequence formed by successively multiplying the initial term by a fixed number called the common ratio.
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Examples:
\( \{a_k\} \) is 1, -1, 1, -1, 1, -1, 1, -1, ...
\( \{b_k\} \) is 1, \( \frac{1}{2} \), \( \frac{1}{4} \), \( \frac{1}{8} \), ...
A **geometric sequence** is a sequence formed by successively multiplying the initial term by a fixed number called the **common ratio**.

**Examples:**

$\{a_k\}$ is 1, -1, 1, -1, 1, -1, 1, -1, ...

$\{b_k\}$ is 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ...

What are the explicit formulas for the above sequences?
A geometric sequence is a sequence formed by successively multiplying the initial term by a fixed number called the common ratio.

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\{a_k\} is 1, -1, 1, -1, 1, -1, 1, -1, ...
\{b_k\} is 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...

What are the explicit formulas for the above sequences?
For any geometric sequence \{s_k\} with initial term \(s_0\) and common ratio \(r\):

\[ s_k = s_0 \cdot r^k, \text{ for } k \geq 0 \]
A **geometric sequence** is a sequence formed by successively multiplying the initial term by a fixed number called the **common ratio**.

**Examples:**

\( \{a_k\} \) is 1, -1, 1, -1, 1, -1, 1, -1, ...

\[ a_k = a_0 \cdot r^k = 1 \cdot (-1)^k \text{ for all } k \geq 0 \]

\( \{b_k\} \) is 1, \( \frac{1}{2} \), \( \frac{1}{4} \), \( \frac{1}{8} \), ...

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**Examples:**

\(
\{a_k\} \text{ is } 1, -1, 1, -1, 1, -1, 1, -1, \ldots
\)

\[
\rightarrow a_k = a_0 \cdot r^k = 1 \cdot (-1)^k \text{ for all } k \geq 0
\]

\[
\{b_k\} \text{ is } 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots
\]

\[
\rightarrow b_k = b_0 \cdot r^k = 1 \cdot \left(\frac{1}{2}\right)^k \text{ for all } k \geq 0
\]

What are the explicit formulas for the above sequences?

For any geometric sequence \(\{s_k\}\) with initial term \(s_0\) and common ratio \(r\):

\[
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Arithmetic Sequences

An arithmetic sequence is a sequence formed by successively adding a fixed number, called the common difference, to the initial term.
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An **arithmetic sequence** is a sequence formed by successively adding a fixed number, called the **common difference**, to the initial term.

**Examples:**

\{a_k\} is 5, 15, 25, 35, 45, ...

\{b_k\} is 49, 42, 35, 28, 21, ...
An **arithmetic sequence** is a sequence formed by successively adding a fixed number, called the **common difference**, to the initial term.

**Examples:**

\( \{a_k\} \) is 5, 15, 25, 35, 45, ...

\( \{b_k\} \) is 49, 42, 35, 28, 21, ...

What are the explicit formulas for the above sequences?
An **arithmetic sequence** is a sequence formed by successively adding a fixed number, called the **common difference**, to the initial term.

**Examples:**

\{a_k\} is 5, 15, 25, 35, 45, ...
\{b_k\} is 49, 42, 35, 28, 21, ...

What are the explicit formulas for the above sequences?

For any arithmetic sequence \{s_k\} with initial term \(s_0\) and common diff \(d\):

\[ s_k = s_0 + kd, \text{ for } k \geq 0 \]
An **arithmetic sequence** is a sequence formed by successively adding a fixed number, called the **common difference**, to the initial term.

**Examples:**

\{a_k\} is 5, 15, 25, 35, 45, ...

\[a_k = a_0 \cdot kd = 5 + 10 \cdot k \text{ for all } k \geq 0\]

\{b_k\} is 49, 42, 35, 28, 21, ...

What are the explicit formulas for the above sequences?

For any arithmetic sequence \{s_k\} with initial term \(s_0\) and common diff \(d\):

\[s_k = s_0 + kd, \text{ for } k \geq 0\]
Arithmetic Sequences

An **arithmetic sequence** is a sequence formed by successively adding a fixed number, called the **common difference**, to the initial term.

**Examples:**

\[ \{a_k\} \text{ is } 5, 15, 25, 35, 45, \ldots \quad \rightarrow \quad a_k = a_0 \cdot kd = 5 + 10 \cdot k \text{ for all } k \geq 0 \]

\[ \{b_k\} \text{ is } 49, 42, 35, 28, 21, \ldots \quad \rightarrow \quad b_k = b_0 \cdot kd = 49 + (-7)k \text{ for all } k \geq 0 \]

What are the explicit formulas for the above sequences?

For any arithmetic sequence \( \{s_k\} \) with initial term \( s_0 \) and common diff \( d \):

\[ s_k = s_0 + kd, \text{ for } k \geq 0 \]
Outline

- Sequences
- **Summations**
- Recurrence Relations
- Mathematical Induction
- Strong Induction
Summation notation is used to express the sum of terms in a numerical sequence.

Consider the sequence: $a_0, a_1, a_2, a_3, ..., a_k$

We can express the sum of all elements in the sequence as:

$$\sum_{i=0}^{k} a_i$$
Summations

**Summation notation** is used to express the sum of terms in a numerical sequence.

Consider the sequence: \( a_0, a_1, a_2, a_3, \ldots, a_k \)

We can express the sum of all elements in the sequence as:

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Summations

**Summation notation** is used to express the sum of terms in a numerical sequence.

Consider the sequence: $a_0, a_1, a_2, a_3, \ldots, a_k$

We can express the sum of all elements in the sequence as: $$\sum_{i=0}^{k} a_i$$

What this represents is: $$\sum_{i=0}^{k} a_i = a_0 + a_1 + a_2 + \ldots + a_k$$
Summations

Given:
\[ \sum_{i=0}^{k} a_i = a_0 + a_1 + a_2 + \ldots + a_k \]

- \( i \) is the **index** of the summation
- \( i = 0 \) and \( i = k \) are the **limits** of the summation
  - 0 is the **lower limit**
  - \( k \) is the **upper limit**
- The capital sigma (\( \Sigma \)) denotes that elements will be added together
Summation Examples

Given $a_i = i$ what does the following sum evaluate to:

$$
\sum_{i=1}^{5} a_i
$$
Summation Examples

Given $a_i = i$ what does the following sum evaluate to:

$$\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5$$
Summation Examples

Given $a_i = i$ what does the following sum evaluate to:

$$\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 2 + 3 + 4 + 5 = 15$$
Summation Examples

Given $b_i = 2^{-i}$ what does the following sum evaluate to:

$$
\sum_{i=0}^{2} b_i
$$
Summation Examples

Given $b_i = 2^i$ what does the following sum evaluate to:

$$\sum_{i=0}^{2} b_i = b_0 + b_1 + b_2$$
Summation Examples

Given $b_i = 2^{-i}$ what does the following sum evaluate to:

$$
\sum_{i=0}^{2} b_i = b_0 + b_1 + b_2 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}
$$
Given \( c_i = i \), what does the following sum evaluate to:

\[
\sum_{i=1}^{n} C_i
\]
Given $c_i = i$, what does the following sum evaluate to:

$$\sum_{i=1}^{n} c_i = c_1 + c_2 + c_3 + \ldots + c_n$$
An Interesting (and useful) Example

Given \( c_i = i \), what does the following sum evaluate to:

\[
\sum_{i=1}^{n} c_i = 1 + 2 + 3 + \ldots + (n - 2) + (n - 1) + n
\]
An Interesting (and useful) Example

Given $c_i = i$, what does the following sum evaluate to:

$$\sum_{i=1}^{n} c_i = 1 + 2 + 3 + \ldots + (n - 2) + (n - 1) + n$$

Idea: What if we rearrange the order (since addition is commutative), and consider pairs of numbers?
An Interesting (and useful) Example

Given $c_i = i$, what does the following sum evaluate to:

$$\sum_{i=1}^{n} c_i = (1 + n) + (2 + (n - 1)) + (3 + (n - 2)) + \ldots + \left( \frac{n}{2} + \left( \frac{n}{2} + 1 \right) \right)$$

Idea: What if we rearrange the order (since addition is commutative), and consider pairs of numbers?
An Interesting (and useful) Example

Given $c_i = i$, what does the following sum evaluate to:

$$\sum_{i=1}^{n} c_i = (1 + n) + (2 + (n - 1)) + (3 + (n - 2)) + \ldots + \left(\frac{n}{2} + \left(\frac{n}{2} + 1\right)\right)$$

Each pair sums to $n + 1$...how many pairs are there?

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An Interesting (and useful) Example

Given $c_i = i$, what does the following sum evaluate to:

$$\sum_{i=1}^{n} c_i = (1 + n) + (2 + (n - 1)) + (3 + (n - 2)) + \ldots + \left(\frac{n}{2} + \left(\frac{n}{2} + 1\right)\right)$$

Each pair sums to $n + 1$...how many pairs are there? $n/2$ pairs

Idea: What if we rearrange the order (since addition is commutative), and consider pairs of numbers?
Given $c_i = i$, what does the following sum evaluate to:

$$\sum_{i=1}^{n} c_i = \frac{(n + 1)n}{2}$$
Outline

- Sequences
- Summations
- **Recurrence Relations**
- Mathematical Induction
- Strong Induction
Idea: Some sequences can naturally be defined as a function of earlier terms in the sequence (defined recursively).
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Example: Consider the factorial function, fact(n) = n * n-1 * n-2 * ... * 1, (also denoted n!)
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Example: Consider the factorial function, $\text{fact}(n) = n \times n-1 \times n-2 \times \ldots \times 1$, (also denoted $n!$)

Notice that $\text{fact}(n) = n \times \text{fact}(n-1)$
**Idea:** Some sequences can naturally be defined as a function of earlier terms in the sequence (defined recursively).

**Example:** Consider the factorial function, $\text{fact}(n) = n \times n-1 \times n-2 \times \ldots \times 1$, (also denoted $n!$)

Notice that $\text{fact}(n) = n \times \text{fact}(n-1)$

...and $\text{fact}(n-1) = n-1 \times \text{fact}(n-2)$
Sequences: Recurrence Relations

**Idea:** Some sequences can naturally be defined as a function of earlier terms in the sequence (defined recursively).

**Example:** Consider the factorial function, \( \text{fact}(n) = n \times (n-1) \times (n-2) \times \ldots \times 1 \), (also denoted \( n! \))

Notice that \( \text{fact}(n) = n \times \text{fact}(n-1) \)

...and \( \text{fact}(n-1) = (n-1) \times \text{fact}(n-2) \)

...etc
A **recurrence relation** for the sequence \( \{a_n\} \) is an equation that expresses \( a_n \) in terms of one or more of the previous terms \((a_0, a_1, a_2, \ldots a_{n-1})\), for all integers \( n \) with \( n \geq n_0 \), where \( n_0 \) is a nonnegative integer.

- A recurrence relation is said to **recursively define** a sequence
- It may have one or more initial terms
- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation
Recurrence Relation Examples

We've already seen some sequences that can be easily defined recursively. 

**Recall**: The geometric sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

**Explicit Formula**: $a_k = \left(\frac{1}{2}\right)^k$ for all $k \geq 0$

**Recurrence Relation:**
Recurrence Relation Examples

We've already seen some sequences that can be easily defined recursively.

Recall: The geometric sequence 1, ½, ¼, ⅛, ...

Explicit Formula: $a_k = \left(\frac{1}{2}\right)^k$ for all $k \geq 0$

Recurrence Relation:

$a_0 = 1$
Recurrence Relation Examples

We've already seen some sequences that can be easily defined recursively.

**Recall:** The geometric sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$

**Explicit Formula:** $a_k = (\frac{1}{2})^k$ for all $k \geq 0$

**Recurrence Relation:**

$a_0 = 1$

$a_k = \frac{1}{2} a_{k-1}$ for $k \geq 1$
Recurrence Relation Examples

We've already seen some sequences that can be easily defined recursively.

**Recall:** The arithmetic sequence 5, 15, 25, 35, ...

**Explicit Formula:** $a_k = 5 + 10k$ for all $k \geq 0$

**Recurrence Relation:**
Recurrence Relation Examples

We've already seen some sequences that can be easily defined recursively

**Recall**: The arithmetic sequence 5, 15, 25, 35, ...

**Explicit Formula**: \( a_k = 5 + 10k \) for all \( k \geq 0 \)

**Recurrence Relation**:
\( a_0 = 5 \)
We've already seen some sequences that can be easily defined recursively.

**Recall**: The arithmetic sequence 5, 15, 25, 35, ...

**Explicit Formula**: \( a_k = 5 + 10k \) for all \( k \geq 0 \)

**Recurrence Relation**:
\[
\begin{align*}
a_0 &= 5 \\
\end{align*}
\]

\[
\begin{align*}
a_k &= 10 + a_{k-1} \quad \text{for } k \geq 1
\end{align*}
\]
The Fibonacci sequence is a very famous recurrence relation
- Begins with **two** initial terms
- Each following term is the sum of the previous 2

First few terms are 0, 1, 1, 2, 3, 5, 8, 13, ...

**Recurrence Relation:**

\[ f_0 = 0 \]

\[ f_1 = 1 \]

\[ f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2 \]

Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} - a_{n-2} \) for \( n \geq 2 \), and suppose that \( a_0 = 3 \) and \( a_1 = 5 \). What are \( a_2 \) and \( a_3 \)?
Recurrence Relation Examples

Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation
\[
a_n = a_{n-1} - a_{n-2}
\]
for \( n \geq 2 \), and suppose that \( a_0 = 3 \) and \( a_1 = 5 \). What are \( a_2 \) and \( a_3 \)?

\[
a_2 = a_{2-1} - a_{2-2} = a_1 - a_0 = 5 - 3 = 2
\]
Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} - a_{n-2} \) for \( n \geq 2 \), and suppose that \( a_0 = 3 \) and \( a_1 = 5 \). What are \( a_2 \) and \( a_3 \)?

\[
a_2 = a_{2-1} - a_{2-2} = a_1 - a_0 = 5 - 3 = 2
\]

\[
a_3 = a_{3-1} - a_{3-2} = a_2 - a_1 = 2 - 5 = -3
\]
Recurrence Relation Examples

Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} + 3 \) for \( n \geq 1 \), and suppose that \( a_0 = 2 \). What are \( a_1 \), \( a_2 \), and \( a_3 \)?
Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} + 3 \) for \( n \geq 1 \), and suppose that \( a_0 = 2 \). What are \( a_1, a_2, \) and \( a_3 \)?

\[
\begin{align*}
a_1 &= a_{1-1} + 3 = a_0 + 3 = 2 + 3 = 5
\end{align*}
\]
Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} + 3 \) for \( n \geq 1 \), and suppose that \( a_0 = 2 \). What are \( a_1 \), \( a_2 \), and \( a_3 \)?

\[
a_1 = a_{1-1} + 3 = a_0 + 3 = 2 + 3 = 5
\]

\[
a_2 = a_{2-1} + 3 = a_1 + 3 = 5 + 3 = 8
\]
Recurrence Relation Examples

Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} + 3 \) for \( n \geq 1 \), and suppose that \( a_0 = 2 \). What are \( a_1 \), \( a_2 \), and \( a_3 \)?

\[
a_1 = a_{1-1} + 3 = a_0 + 3 = 2 + 3 = 5
\]

\[
a_2 = a_{2-1} + 3 = a_1 + 3 = 5 + 3 = 8
\]

\[
a_3 = a_{3-1} + 3 = a_2 + 3 = 8 + 3 = 11
\]
How do we solve a recurrence relation to find an explicit formula for $a_n$?

Two approaches:

1. Start at the initial condition and successively apply the recurrence relation upwards until you reach $a_n$ to deduce a formula.
2. Start with the term $a_n$ and work downward until you reach the initial condition and deduce a formula.
Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} + 3 \) for \( n \geq 2 \), and suppose that \( a_1 = 2 \). Find the formula for \( a_n \).
Solving Recurrence Relations

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n \geq 2$, and suppose that $a_1 = 2$. Find the formula for $a_n$.

\[
\begin{align*}
a_1 & = 2 \\
a_2 & = a_{2-1} + 3 = a_1 + 3 = 2 + 3 = 5 \\
a_3 & = a_{3-1} + 3 = (2+3) + 3 = 2 + 2 \times 3 \\
a_4 & = a_{4-1} + 3 = (2+2 \times 3) + 3 = 2 + 3 \times 3 \\
& \vdots \\
a_n & = 2 + (n-1) \times 3
\end{align*}
\]
# Solving Recurrence Relations

Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} + 3 \) for \( n \geq 2 \), and suppose that \( a_1 = 2 \). Find the formula for \( a_n \).

<table>
<thead>
<tr>
<th>( a_n )</th>
<th>( a_{n-1} + 3 )</th>
<th>( a_{n-2} + 2*3 )</th>
<th>( a_{n-3} + 3*3 )</th>
<th>( ... )</th>
<th>( a_1 + (n-1)*3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 ) = 2</td>
<td>( a_2 = a_1 + 3 = 5 )</td>
<td>( a_3 = a_2 + 3 = 2 + 2*3 )</td>
<td>( a_4 = a_3 + 3 = 2 + 3*3 )</td>
<td>( ... )</td>
<td>( a_n = 2 + (n-1) * 3 )</td>
</tr>
</tbody>
</table>
Outline

- Sequences
- Summations
- Recurrence Relations
- **Mathematical Induction**
- Strong Induction
What is Induction?

Suppose we want to prove a property, $P(n)$, is TRUE for all $n \in \mathbb{N}$ (or $\mathbb{N} \cup \{0\}$):
What is Induction?

Suppose we want to prove a property, $P(n)$, is TRUE for all $n \in \mathbb{N}$ (or $\mathbb{N} \cup \{0\}$):

For example, given an arithmetic sequence with $a_0 = 4$ and common difference $d = 16$, prove that $a_n$ is divisible by 4 for all $n \geq 0$...

- Here the property is the predicate $P(n)$: "$a_n$ is divisible by 4"
- We want to prove $\forall n \in \mathbb{N} \cup \{0\}, P(n)$
What is Induction?

Suppose we want to prove a property, $P(n)$, is TRUE for all $n \in \mathbb{N}$ (or $\mathbb{N} \cup \{0\}$):

Another example, prove that $\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$

- Here the property is the predicate $P(n)$: $\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$
- We want to prove $\forall n \in \mathbb{N}, P(n)$
In both of the previous examples, notice that there is some dependency between truth values $P(k)$ and $P(k + 1)$...
Mathematical Induction

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For the first example: we know that $a_1 = a_0 + d$, $a_2 = a_1 + d$, etc...

- Notice how the formula for $a_1$ relies on $a_0$. Therefore $P(1)$ relies on $P(0)$
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Similarly, consider the values of our summation for upper limit = $k + 1$

\[ \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k + 1) \]
Mathematical Induction

In both of the previous examples, notice that there is some dependency between truth values $P(k)$ and $P(k + 1)$...

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Similarly, consider the values of our summation for upper limit = $k + 1$

$$
\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k + 1)
$$

Mathematical Induction takes advantage of these dependencies
Mathematical Induction

**Idea:** Mathematical Induction consists of two proofs

1. Prove that $P(k_0)$ is a true statement (where $k_0$ is our initial index in $D$)
2. Prove that $\forall k, (P(k) \rightarrow P(k + 1))$ is a true statement

**Conclusion:**

Since $P(k_0)$ is a true statement, and $P(k) \rightarrow P(k + 1)$ it holds that $P(n)$ is true for all $n$ in $D$
Mathematical Induction

**Principle of Mathematical Induction:**

Let $P(n)$ be a statement defined for any $n \in \mathbb{N}$. If the following hold:

- $P(1)$ is true
- For all $k \in \mathbb{N}$, $P(k) \rightarrow P(k + 1)$

Then $P(n)$ is true for all $n \in \mathbb{N}$

**Note:** We can relax this to apply to any domain $D$ of consecutive integers.

To prove the inductive step, assume $P(k)$, then derive $P(k + 1)$.
Example: Towers of Hanoi

Brief Live Demo...
Example: Towers of Hanoi

For the Towers of Hanoi problem, let $P(n)$: We can move $n$ disks.
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To prove that this works for all $n \in \mathbb{N}$, we first show the base case $P(1)$.

Clearly we can move 1 disk, so $P(1)$ is TRUE.
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Example: Towers of Hanoi

For the Towers of Hanoi problem, let \( P(n) \): We can move \( n \) disks.

To prove that this works for all \( n \in \mathbb{N} \), we first show the base case \( P(1) \).

Clearly we can move 1 disk, so \( P(1) \) is TRUE.

Then we assume we can move \( k \) disks for arbitrary \( k \geq 1 \), and show that this implies we can also move \( k + 1 \) disks.

If we can move \( k \), then we can move them to the middle platform, move the one remaining disk to the right platform (because \( P(1) \) is true), then move the \( k \) disks from the middle to the right. So \( P(k) \rightarrow P(k + 1) \).
Proof Template

To formally prove something via induction, you must do all of the following:

1. Express the statement being proved in the form \( \forall n \geq b, P(n) \) for a fixed integer \( b \)
2. Prove the **Base Case**: show that \( P(b) \) is true.
3. Prove the **Inductive Case**
   a. State the inductive hypothesis in the form "assume that \( P(k) \) is true for an arbitrary \( k \geq b \)
   b. State what must be proved under this assumption; write out \( P(k + 1) \)
   c. Prove the statement \( P(k + 1) \) is true by using the assumption \( P(k) \) is true. Be sure this proof is valid for all integers \( k \geq b \)
   d. Clearly identify the conclusion
4. Now that you have proven the Base Case and Inductive Case, conclude that \( P(n) \) is true for all \( n \geq b \)
Proof Example

Prove that if $n$ is a positive integer, 

$$
\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}
$$
Proof Example

Prove that if $n$ is a positive integer, \[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \]

Predicate: $P(n)$: \[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \]

Goal: Show that $P(n)$ is true for all $n \geq 1$
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Proof Example

Prove that if \( n \) is a positive integer, 

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\[
\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}
\]

**Goal:** Show that \( P(n) \) is true for all \( n \geq 1 \)

**Base Case:**

Show \( P(1) \) is true
Proof Example

Prove that if $n$ is a positive integer,\[\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}\]

**Predicate:** $P(n)$: $\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$

**Goal:** Show that $P(n)$ is true for all $n \geq 1$

**Base Case:**

Show $P(1)$ is true

$P(1): 1 = 1 \times (1+1) / 2 \ldots$ TRUE
Proof Example

Prove that if \( n \) is a positive integer, \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \)

**Predicate:** \( P(n) \): \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \)

**Goal:** Show that \( P(n) \) is true for all \( n \geq 1 \)

**Inductive Case:**
Proof Example

Prove that if $n$ is a positive integer, 

$$\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$$

Predicate: $P(n)$: 

$$\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$$

Goal: Show that $P(n)$ is true for all $n \geq 1$

Inductive Case:

Assume that $P(k)$ is true for an arbitrary integer $k \geq 1$
Proof Example

Prove that if $n$ is a positive integer, \[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

**Predicate:** $P(n)$: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

**Goal:** Show that $P(n)$ is true for all $n \geq 1$

**Inductive Case:**

Assume that $P(k)$ is true for an arbitrary integer $k \geq 1$

Now we must derive that $P(k + 1)$ is true...
Proof Example

\[
\sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + (k + 1)
\]
Proof Example

\[
\sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + (k + 1)
\]

This is just the sum from \(i=1\) to \(k\)

\[
\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k + 1)
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Proof Example

\[ \sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + (k + 1) \]

This is just the sum from \( i=1 \) to \( k \)

\[ \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k + 1) \]

We have assumed that \( P(k) \) is true

\[ \sum_{i=1}^{k+1} i = \frac{k(k + 1)}{2} + (k + 1) \]
Proof Example

\[ \sum_{i=1}^{k+1} i = \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2} \]

Therefore if we assume \( P(k) \) is true, then \( P(k + 1) \) must also be true.
Proof Example

Prove that if $n$ is a positive integer, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Predicate: $P(n)$: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Goal: Show that $P(n)$ is true for all $n \geq 1$

**Base Case:** True!

**Inductive Case:** True!

Therefore since the Base Case and Inductive Case are both true, $P(n)$ is true for all $n \geq 1$
Prove that for all $n \geq b$, "..." is true.

$P(n)$: "...

**Base Case:** Show that $P(b)$ is true

**Inductive Case:** We must show that if $P(k)$ is true, then $P(k + 1)$ is true

- **Inductive Hypothesis:** Assume that $P(k)$ is true for an arbitrary positive integer $k \geq b$.
- We must now show that $P(k + 1)$: "..." is true
- Given $P(k + 1) = "..."
  - More derivation...
  - Given our Inductive Hypothesis ....... show that $P(k + 1)$ must be true

Therefore $P(k) \rightarrow P(k + 1)$

Therefore both the Base Case and Inductive case are true, so $P(n)$: "..." is true for all $n \geq b$
Outline

- Sequences
- Summations
- Recurrence Relations
- Mathematical Induction
- **Strong Induction**
Problem: What happens when our next term (at $k + 1$) depends on multiple previous terms (more than just term $k$)?
Strong Induction

**Problem:** What happens when our next term (at $k + 1$) depends on multiple previous terms (more than just term $k$)?

**Example:** The terms in the Fibonacci sequence rely on two previous terms.
Strong Induction

Principle of Strong Mathematical Induction

Let $a, b$ be integers with $a \leq b$

Let $P(n)$ be a statement defined for any integer $n \geq a$

Then $P(n)$ is true for all $n \geq a$ if the following two conditions hold:

1. $P(a), P(a + 1), ..., P(b)$ are all individually true (the base cases)
2. For all $k \geq b$, $P(a) \land P(a + 1) \land ... \land P(k) \rightarrow P(k + 1)$ (the inductive case)
To formally prove something via strong induction, you must do all of the following:

1. Express the statement being proved in the form "$\forall n \geq a, P(n)$ for a fixed integer $a$.
2. Prove the Base Cases: show that $P(a), P(a + 1), \ldots, P(b)$ are all true.
3. Prove the Inductive Case:
   a. Inductive Hypothesis: Assume $P(i)$ is true for all $i$, where $a \leq i \leq k$ for an arbitrary $k \geq b$.
   b. State what must be proved under this assumption; write out $P(k + 1)$.
   c. Prove the statement $P(k + 1)$ is true by using the inductive assumption.
   d. Clearly identify the conclusion.
4. Now that you have proven the Base Case and Inductive Case, conclude that $P(n)$ is true for all $n \geq a$ by the principle of strong mathematical induction.
Strong Mathematical Induction Example

Prove that a class of \( n \geq 12 \) students can be divided into groups of 4 or 5
Strong Mathematical Induction Example

Prove that a class of \( n \geq 12 \) students can be divided into groups of 4 or 5

\( P(n) \): For all \( n \geq 12 \), a class of size \( n \) can be divided into groups of 4 or 5

**Base Cases:**

Prove \( P(n) \) is true for \( n = 12, 13, 14, \) and 15
Prove that a class of \( n \geq 12 \) students can be divided into groups of 4 or 5

\[ P(n) \]: For all \( n \geq 12 \), a class of size \( n \) can be divided into groups of 4 or 5

**Base Cases:**

Prove \( P(n) \) is true for \( n = 12, 13, 14, \) and \( 15 \)

\[
\begin{align*}
12 &= 4 + 4 + 4 \\
13 &= 4 + 4 + 5 \\
14 &= 4 + 5 + 5 \\
15 &= 5 + 5 + 5
\end{align*}
\]
Prove that a class of $n \geq 12$ students can be divided into groups of 4 or 5

$P(n)$: For all $n \geq 12$, a class of size $n$ can be divided into groups of 4 or 5

**Base Cases:**
Prove $P(n)$ is true for $n = 12, 13, 14,$ and $15$

12 = 4 + 4 + 4
13 = 4 + 4 + 5
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Therefore $P(12), P(13), P(14),$ and $P(15)$ are all TRUE
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Prove that a class of $n \geq 12$ students can be divided into groups of 4 or 5

$P(n)$: For all $n \geq 12$, a class of size $n$ can be divided into groups of 4 or 5

**Inductive Case:**

We must show that if $P(i)$ is true for $12 \leq i \leq k$, where $k \geq 15$, then $P(k + 1)$ must be true.
Strong Mathematical Induction Example

Prove that a class of $n \geq 12$ students can be divided into groups of 4 or 5

$P(n)$: For all $n \geq 12$, a class of size $n$ can be divided into groups of 4 or 5

**Inductive Case:**

We must show that if $P(i)$ is true for $12 \leq i \leq k$, where $k \geq 15$, then $P(k + 1)$ must be true.

**Inductive Hypothesis:** Let $k \geq 15$

Assume $P(i)$ is true for all $i$, where $12 \leq i \leq k$. Notably, $P(k - 3)$ is true...
Proof of the Inductive Case:

We must show that \( P(k + 1) \) is true.

From the \( k + 1 \) students, form a group of 4 students. Now \((k + 1) - 4 = k - 3\) students remain to be grouped.

By our inductive hypothesis \( P(k - 3) \) is true, so we can divide the remaining \( k - 3 \) students into groups of 4 or 5.

Therefore if \( P(i) \) is true for all \( i \), where \( 12 \leq i \leq k \), then \( P(k + 1) \) must be true.
Prove that a class of $n \geq 12$ students can be divided into groups of 4 or 5

$P(n)$: For all $n \geq 12$, a class of size $n$ can be divided into groups of 4 or 5

**Base Cases:** TRUE

**Inductive Case:** TRUE

Therefore by the principle of strong mathematical induction, we can conclude that $P(n)$ is true for all $n \geq 12$
Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).
Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

Do we need strong mathematical induction here?
Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \)

Do we need strong mathematical induction here? Yes. The later terms in our sequence rely on more than just the previous term.
One Last Induction Example

Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

1. Express the statement being proved in the form "\( \forall n \geq a, P(n) \) for a fixed integer \( a \)."
2. Prove the **Base Cases**: show that \( P(a), P(a+1), \ldots, P(b) \) are all true.
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   a. Inductive Hypothesis: Assume \( P(i) \) is true for all \( i \), where \( a \leq i \leq k \) for an arbitrary \( k \geq b \).
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   d. Clearly identify the conclusion.
4. Now that you have proven the Base Case and Inductive Case, conclude that \( P(n) \) is true for all \( n \geq a \) by the principle of strong mathematical induction.
One Last Induction Example

Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

1. Express the statement being proved in the form " \( \forall n \geq a, P(n) \) for a fixed integer \( a \)"

\[ P(n): a_n = 3 \cdot 2^{n-1} + 2(-1)^n, \] we will now show that for all \( n \in \mathbb{N}, P(n) \) is true.
One Last Induction Example

Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

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One Last Induction Example

Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, a_2 = 8, a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \). Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

2. Prove the **Base Cases**: show that \( P(a), P(a + 1), \ldots, P(b) \) are all true.

\[ P(1): a_1 = 1, a_1 = 3 \cdot 2^{n-1} + 2(-1)^n = 3 \cdot 2^0 + 2(-1)^1 = 3 - 2 = 1 \; \checkmark \]

\[ P(2): a_2 = 8, a_2 = 3 \cdot 2^{n-1} + 2(-1)^n = 3 \cdot 2^1 + 2(-1)^2 = 6 + 2 = 8 \; \checkmark \]

Therefore our base cases, \( P(1) \) and \( P(2) \) are both true.
Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \; a_2 = 8, \; a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

1. Express the statement being proved in the form "\( \forall n \geq a, P(n) \) for a fixed integer \( a \)."
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Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

3. Prove the Inductive Case
   a. **Inductive Hypothesis:** Assume \( P(i) \) is true for all \( i \), where \( a \leq i \leq k \) for an arbitrary \( k \geq b \).

   Assume \( P(i): a_i = 3 \cdot 2^{i-1} + 2(-1)^i \) for all \( 1 \leq i \leq k \), where \( k \geq 2 \)
Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

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One Last Induction Example

Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

b. State what must be proved under this assumption; write out \( P(k+1) \)

Now we must prove that \( P(k+1) \) is true. Specifically, we must show that \( a_{k+1} = 3 \cdot 2^{k+1-1} + 2(-1)^{k+1} \).
Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

c. Prove the statement \( P(k + 1) \) is true by using the inductive assumption.

We must show \( a_{k+1} = 3 \cdot 2^{k+1-1} + 2(-1)^{k+1} \)

\[
a_{k+1} = a_k + 2a_{k-1} \quad \text{(by definition of our recurrence relation)}
\]

\[
a_{k+1} = 3 \cdot 2^{k-1} + 2(-1)^k + 2(3 \cdot 2^{k-2} + 2(-1)^{k-2}) \quad \text{(by subbing in our assumption)}
\]

\[
= 3(2^{k-1} + 2^{k-1}) = 2((-1)^k + 2(-1)^{k-1}) = 3 \cdot 2^k + 2(-1)^{k+1}
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Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, a_2 = 8, a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

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One Last Induction Example

Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

1. Express the statement being proved in the form " \( \forall n \geq a, P(n) \) for a fixed integer \( a \).
2. Prove the Base Cases: show that \( P(a), P(a + 1), \ldots, P(b) \) are all true.
3. Prove the Inductive Case
   a. Inductive Hypothesis: Assume \( P(i) \) is true for all \( i \), where \( a \leq i \leq k \) for an arbitrary \( k \geq b \)
   b. State what must be proved under this assumption; write out \( P(k + 1) \)
   c. Prove the statement \( P(k + 1) \) is true by using the inductive assumption
   d. Clearly identify the conclusion
4. Now that you have proven the Base Case and Inductive Case, conclude that \( P(n) \) is true for all \( n \geq a \) by the principle of strong mathematical induction.
One Last Induction Example

Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \). Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

d. Clearly identify the conclusion

Therefore \( P(k+1) \) is true when \( P(i) \) is true for all \( 1 \leq i \leq k \), where \( k \geq 2 \).

So the proof of the induction step is complete.
One Last Induction Example

Let \( \{a_n\} \) be the sequence defined by \( a_1 = 1, \ a_2 = 8, \ a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all \( n \in \mathbb{N} \).

1. Express the statement being proved in the form "\( \forall n \geq a, P(n) \) for a fixed integer \( a \)."
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By the principle of strong mathematical induction, \( P(n) \) is true for all \( n \in \mathbb{N} \).