CSE 191

Introduction to Discrete Structures

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Sequences, Summations, and Induction

Mid-Semester Review

Topics Covered so Far…

1. Propositional and Predicate Logic
2. Logical and Mathematical Proofs
3. Sets
4. Functions and Relations

Outline

- **Sequences**
  - **Definition and Terminology**
  - Finite Sequences
  - Infinite Sequences
  - Explicit Formulas
- Summations
- Recurrence Relations
- Mathematical Induction
- Strong Induction

Sequences

Example: Let's say you want to represent your GPA over four semester…

Consider the function g: {1,2,3,4} → { gpa | gpa ∈ ℝ, 0 ≤ gpa ≤ 4 } defined by:

$g(1) = 3.6$ $g(2) = 2.8$ $g(3) = 3.2$ $g(4) = 3.8$

For shorthand, we write $a(n)$ as a_n

So g_1, g_2, g_3, g_4 represents our GPA **sequence**, 3.6, 2.8, 3.2, 3.8

Sequences: Terminology

A ***sequence*** is created by a special type of function with a domain of consecutive integers…ie no gaps in the domain

OK: $\mathbb{N}$, $\mathbb{Z}^+$, $\mathbb{Z}^+ \cup \{0\}$, $\mathbb{Z}$, {1,2,34,5}

Not OK: {1,3,5,7}, $\{ x \in \mathbb{N} \mid x \text{ is even} \}$

Sequences: Terminology

Given a sequence, ***a*** over domain $D \subseteq \mathbb{Z}$

- A **single term** (or **term**) ***a***(***k***) is written as a_k for any $k \in D$
- Given term a_k, ***k*** denotes the **index** of a_k
- A shorthand for the entire sequence is $\{ a_k \}$ or $\{ a_n \}$

Sequences: Ground Rules

- Most sequences start at index 0 or 1
  - The domain is typically ℕ or ℕ ∪ {0}
  - In this case we write $a_1, a_2, a_3, \ldots$ or $b_0, b_1, b_2, \ldots$
- Sequences **can** start from any integer
  - A sequence starting at -2: $a_{-2}, a_{-1}, a_0, a_1, \ldots$
  - A sequence starting at 5: $b_5, b_6, b_7, b_8, \ldots$
- The term b_8 might not necessarily be the 8th term
  - In $b_5, b_6, b_7, b_8, \ldots$ the term b_8 is the 4th term

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- **Sequences**
    - Definition and Terminology
    - **Finite Sequences**
    - Infinite Sequences
    - Explicit Formulas
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Sequences: Finite

A sequence with a finite domain is called a **finite sequence**.

Consider $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$

There is an **initial index** m and a_m denotes the **initial term**

There is a **final index** n and a_n denotes the **final term**

Examples:

GPA over 4 semesters

1,2,3,4

5,4,5,4,5,4,5

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Sequences: Infinite

A sequence with an infinite domain is called an **infinite sequence**.

Consider …, a_m, a_{m+1}, a_{m+2}, …

There may or may not be an **initial index** m (and **initial term** a_m)

There may or may not be an **final index** n (and **final term** a_n)

Examples:

1,3,5,7, … (positive odd numbers)

2,4,6,8, … (positive even numbers)

{1}, {1,2}, {1,2,3}, {1,2,3,4}, … (the sets A_i we had previously defined)

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Sequences: Explicit

An **explicit formula** lets us compute the value of term $\boldsymbol{a_k}$ as a function of $\boldsymbol{k}$

Examples:

$c_k = 5$ for $1 \le k \le 5$ $\rightarrow \{ c_k \} = 5,5,5,5,5$

$d_k = k$ for $1 \le k \le 10$ $\rightarrow \{ d_k \} = 1,2,3,4,5,6,7,8,9,10$

$e_k = 2k$ for $k \ge 1$ $\rightarrow \{ e_k \} = 2,4,6,8, \ldots$

$f_k = 2^k$ for $k \ge 0$ $\rightarrow \{ f_k \} = 1,2,4,8,16, \ldots$

Increasing Sequences

A sequence $\{a_k\}$ is **increasing** if, $\forall i, a_i < a_{i+1}$

A sequence $\{a_k\}$ is **non-decreasing** if, $\forall i, a_i \leq a_{i+1}$

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What about the sequence $\{ h_k \} = 1, 2, 2, 2, 3$?

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What about the sequence $\{ h_k \} = 1, 2, 2, 2, 3$? **Non-Decreasing**

Every increasing sequence is non-decreasing

Not every non-decreasing sequence is increasing

Decreasing Sequences

A sequence $\{a_k\}$ is **decreasing** if, $\forall i, a_i > a_{i+1}$

A sequence $\{a_k\}$ is **non-increasing** if, $\forall i, a_i \geq a_{i+1}$

Decreasing Sequences

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A sequence $\{a_k\}$ is **non-increasing** if, $\forall i, a_i \geq a_{i+1}$

$s_k = 10 - k$ for $1 \leq k \leq 10$ is **decreasing (and non-increasing)**

$t_k = -2k$ for $k \geq 1$ is **decreasing (and non-increasing)**

$u_k = 2^{-k}$ for $k \geq 0$ is **decreasing (and non-increasing)**

What about the sequence $\{vh_k\} = 3, 2, 2, 2, 1$? **Non-Increasing**

Geometric Sequences

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$\{b_k\}$ is 1, ½, ¼, ⅛, …

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For any geometric sequence $\{s_k\}$ with initial term s_0 and common ratio r:

$$s_k = s_0 \cdot r^k, \text{ for } k \geq 0$$

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$\{a_k\}$ is 1, -1, 1, -1, 1, -1, 1, -1, … $\rightarrow a_k = a_0 \cdot r^k = 1 \cdot (-1)^k$ for all $k \geq 0$

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$\{b_k\}$ is 1, ½, ¼, ⅛, … $\rightarrow b_k = b_0 \cdot r^k = 1 \cdot (½)^k$ for all $k \geq 0$

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$\{\boldsymbol{a_k}\}$ is 5, 15, 25, 35, 45, …

$\{\boldsymbol{b_k}\}$ is 49, 42, 35, 28, 21, …

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$\{a_k\}$ is 5, 15, 25, 35, 45, … → $a_k = a_0 \cdot kd = 5 + 10 \cdot k$ for all $k \geq 0$

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$\{b_k\}$ is 49, 42, 35, 28, 21, … → $b_k = b_0 \cdot kd = 49 + (-7)k$ for all $k \geq 0$

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- Sequences
- **Summations**
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Summations

Summation notation is used to express the sum of terms in a numerical sequence

Consider the sequence: $a_0, a_1, a_2, a_3, \ldots, a_k$

We can express the sum of all elements in the sequence as: $$\sum_{i=0}^{k} a_i$$

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What this represents is: $$\sum_{i=0}^{k} a_i = a_0 + a_1 + a_2 + \ldots + a_k$$

Summations

Given: $$\sum_{i=0}^{k} a_i = a_0 + a_1 + a_2 + ... + a_k$$

- ***i*** is the **index** of the summation
- ***i* = 0** and ***i* = *k*** are the **limits** of the summation
  - **0** is the **lower limit**
  - ***k*** is the **upper limit**
- The capital sigma (Σ) denotes that elements will be added together

Summation Examples

Given $\boldsymbol{a_i} = \boldsymbol{i}$ what does the following sum evaluate to:

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Given $\boldsymbol{a_i} = \boldsymbol{i}$ what does the following sum evaluate to:

$$\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 2 + 3 + 4 + 5 = 15$$

Summation Examples

Given $\boldsymbol{b_i} = 2^{-i}$ what does the following sum evaluate to:

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Summation Examples

Given $\boldsymbol{b_i} = 2^{-i}$ what does the following sum evaluate to:

$$\sum_{i=0}^{2} b_i = b_0 + b_1 + b_2 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

An Interesting (and useful) Example

Given $c_i = i$, what does the following sum evaluate to:

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Given $\boldsymbol{c_i} = \boldsymbol{i}$, what does the following sum evaluate to:

$$\sum_{i=1}^{n} c_i = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$

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Idea: What if we rearrange the order (since addition is commutative), and consider pairs of numbers?

An Interesting (and useful) Example

Given $c_i = i$, what does the following sum evaluate to:

$$\sum_{i=1}^{n} c_i = (1 + n) + (2 + (n - 1)) + (3 + (n - 2)) + \ldots + (\frac{n}{2} + (\frac{n}{2} + 1))$$

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Each pair sums to n + 1…how many pairs are there?

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Each pair sums to n + 1…how many pairs are there? n/2 pairs

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An Interesting (and useful) Example

Given $\boldsymbol{c_i} = \boldsymbol{i}$, what does the following sum evaluate to:

$$\sum_{i=1}^{n} c_i = \frac{(n+1)n}{2}$$

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Sequences: Recurrence Relations

Idea: Some sequences can naturally be defined as a function of earlier terms in the sequence (defined recursively).

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Notice that fact(n) = n * fact(n-1)

…and fact(n-1) = n-1 * fact(n-2)

…etc

Sequences: Recurrence Relations

A <u>**recurrence relation**</u> for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms (a_0, a_1, a_2, … a_{n-1}), for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A recurrence relation is said to ***recursively define*** a sequence
- It may have one or more initial terms
- A sequence is called a ***solution*** of a recurrence relation if its terms satisfy the recurrence relation

Recurrence Relation Examples

We've already seen some sequences that can be easily defined recursively

Recall: The geometric sequence 1, ½, ¼, ⅛, …

Explicit Formula: $a_k = (½)^k$ for all $k \geq 0$

Recurrence Relation:

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<u>Recurrence Relation:</u>

$a_0 = 1$

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Explicit Formula: $a_k = (½)^k$ for all $k \geq 0$

Recurrence Relation:

$a_0 = 1$

$a_k = ½\, a_{k-1}$ for $k \geq 1$

Recurrence Relation Examples

We've already seen some sequences that can be easily defined recursively

Recall: The arithmetic sequence 5, 15, 25, 35, …

Explicit Formula: $a_k = 5 + 10k$ for all $k \geq 0$

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Recurrence Relation Examples

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Explicit Formula: $a_k = 5 + 10k$ for all $k \geq 0$

Recurrence Relation:

$a_0 = 5$

Recurrence Relation Examples

We've already seen some sequences that can be easily defined recursively

Recall: The arithmetic sequence 5, 15, 25, 35, …

Explicit Formula: $a_k = 5 + 10k$ for all $k \geq 0$

Recurrence Relation:

$a_0 = 5$

$a_k = 10 + a_{k-1}$ for $k \geq 1$

Fibonacci

The Fibonacci sequence is a very famous recurrence relation

- Begins with **two** initial terms
- Each following term is the sum of the previous 2

First few terms are 0, 1, 1, 2, 3, 5, 8, 13, …

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibBio.html

Recurrence Relation:

$f_0 = 0$

$f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ **for** $n \geq 2$

Recurrence Relation Examples

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Recurrence Relation Examples

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n ≥ 2, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

$a_2 = a_{2-1} - a_{2-2} = a_1 - a_0 = 5 - 3 = 2$

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$$a_2 = a_{2-1} - a_{2-2} = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_{3-1} - a_{3-2} = a_2 - a_1 = 2 - 5 = -3$$

Recurrence Relation Examples

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n \geq 1$, and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

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$a_1 = a_{1-1} + 3 = a_0 + 3 = 2 + 3 = 5$

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$a_1 = a_{1-1} + 3 = a_0 + 3 = 2 + 3 = 5$

$a_2 = a_{2-1} + 3 = a_1 + 3 = 5 + 3 = 8$

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$a_2 = a_{2-1} + 3 = a_1 + 3 = 5 + 3 = 8$

$a_3 = a_{3-1} + 3 = a_2 + 3 = 8 + 3 = 11$

Solving Recurrence Relations

How do we solve a recurrence relation to find an explicit formula for a_n?

Two approaches:

1. Start at the initial condition and successively apply the recurrence relation upwards until you reach a_n to deduce a formula.
2. Start with the term a_n and work downward until you reach the initial condition and deduce a formula.

Solving Recurrence Relations

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$a_1 = 2$

$a_2 = a_{2-1} + 3 = a_1 + 3 = 2 + 3 = 5$

$a_3 = a_{3-1} + 3 = (2+3)+3 = 2 + 2*3$

$a_4 = a_{4-1} + 3 = (2+2*3)+3 = 2 + 3*3$

…

$a_n = 2 + (n-1) * 3$

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$a_4 = a_{4-1} + 3 = (2+2*3)+3 = 2 + 3*3$

…

$a_n = 2 + (n-1) * 3$

$a_n = a_{n-1} + 3$

$= (a_{n-2} + 3) + 3$

$= a_{n-2} + 2*3$

$= (a_{n-3} + 3) + 2*3$

$= a_{n-3} + 3*3$

…

$= a_1 + (n - 1)*3 = 2 + (n-1)*3$

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- Sequences
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- **Mathematical Induction**
- Strong Induction

What is Induction?

Suppose we want to prove a property, $P(n)$, is TRUE for all $n \in \mathbb{N}$ (or $\mathbb{N} \cup \{0\}$):

What is Induction?

Suppose we want to prove a property, *P*(*n*), is TRUE for all *n* ∈ ℕ (or ℕ ∪ {0}):

For example, given an arithmetic sequence with a_0 = 4 and common difference ***d*** = 16, prove that a_n is divisible by 4 for all ***n*** ≥ 0…

- Here the property is the predicate ***P*(*n*):** "a_n is divisible by 4"
- We want to prove **∀*n*** ∈ ℕ ∪ **{0}, *P*(*n*)**

What is Induction?

Suppose we want to prove a property, *P*(*n*), is TRUE for all *n* $\in \mathbb{N}$ (or $\mathbb{N} \cup \{0\}$):

Another example, prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- Here the property is the predicate ***P*(*n*):** $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- We want to prove **$\forall n \in \mathbb{N}, P(n)$**

Mathematical Induction

In both of the previous examples, notice that there is some dependency between truth values $\boldsymbol{P(k)}$ and $\boldsymbol{P(k + 1)}$...

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For the first example: we know that $a_1 = a_0 + d$. $a_2 = a_1 + d$, etc…

- Notice how the formula for a_1 relies on a_0. Therefore $P(1)$ relies on $P(0)$

Mathematical Induction

In both of the previous examples, notice that there is some dependency between truth values ***P*(*k*)** and ***P*(*k* + 1)...**

For the first example: we know that $\boldsymbol{a_1} = \boldsymbol{a_0} + \boldsymbol{d}$. $\boldsymbol{a_2} = \boldsymbol{a_1} + \boldsymbol{d}$, etc...

- Notice how the formula for $\boldsymbol{a_1}$ relies on $\boldsymbol{a_0}$. Therefore ***P***(1) relies on ***P***(0)

Similarly, consider the values of our summation for upper limit = ***k*** + 1

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$

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Mathematical Induction takes advantage of these dependencies

Mathematical Induction

Idea: Mathematical Induction consists of two proofs

1. Prove that $P(k_0)$ is a true statement (where k_0 is our initial index in D)
2. Prove that $\forall k, (P(k) \rightarrow P(k + 1))$ is a true statement

Conclusion:

Since $P(k_0)$ is a true statement, and $P(k) \rightarrow P(k + 1)$ it holds that $P(n)$ is true for all n in D

Mathematical Induction

Principle of Mathematical Induction:

Let $P(n)$ be a statement defined for any $n \in \mathbb{N}$. If the following hold:

- $P(1)$ is true
- For all $k \in \mathbb{N}$, $P(k) \to P(k + 1)$

Then $P(n)$ is true for all $n \in \mathbb{N}$

Note: We can relax this to apply to any domain D of consecutive integers

To prove the inductive step, assume $P(k)$, then derive $P(k + 1)$

Example: Towers of Hanoi

Brief Live Demo…

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Clearly we can move 1 disk, so ***P*(1)** is TRUE

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To prove that this works for all ***n*** $\in \mathbb{N}$, we first show the base case ***P*(1).**

Clearly we can move 1 disk, so ***P*(1)** is TRUE

Then we assume we can move ***k*** disks for arbitrary ***k*** ≥ 1, and show that this implies we can also move ***k* + 1** disks

If we can move ***k***, then we can move them to the middle platform, move the one remaining disk to the right platform (because ***P*(1)** is true), then move the ***k*** disks from the middle to the right. So ***P*(*k*)** → ***P*(*k* + 1)**

Proof Template

To formally prove something via induction, you must do **all** of the following:

1. Express the statement being proved in the form " $\forall n \geq b, P(n)$ for a fixed integer b
2. Prove the **Base Case**: show that $P(b)$ is true.
3. Prove the **Inductive Case**
   a. State the inductive hypothesis in the form "assume that $P(k)$ is true for an arbitrary $k \geq b$
   b. State what must be proved under this assumption; write out $P(k + 1)$
   c. Prove the statement $P(k + 1)$ is true by using the assumption $P(k)$ is true. Be sure this proof is valid for all integers $k \geq b$
   d. Clearly identify the conclusion
4. Now that you have proven the Base Case and Inductive Case, conclude that $P(n)$ is true for all $n \geq b$

Proof Example

Prove that if ***n*** is a positive integer, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

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Goal: Show that ***P*(*n*)** is true for all ***n* ≥ 1**

Base Case:

Show ***P*(1)** is true

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Prove that if ***n*** is a positive integer, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

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Goal: Show that ***P*(*n*)** is true for all ***n* ≥ 1**

<u>Base Case:</u>

Show ***P*(1)** is true

***P*(1): 1 = 1 * (1+1) / 2… TRUE**

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<u>**Inductive Case:**</u>

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Predicate: ***P*(*n*)**: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Goal: Show that ***P*(*n*)** is true for all ***n* ≥ 1**

<u>Inductive Case:</u>

Assume that ***P*(k)** is true for an arbitrary integer ***k* ≥ 1**

Proof Example

Prove that if ***n*** is a positive integer, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Predicate: *P*(*n*): $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Goal: Show that ***P*(*n*)** is true for all ***n* ≥ 1**

<u>Inductive Case:</u>

Assume that ***P*(k)** is true for an arbitrary integer ***k* ≥ 1**

Now we must derive that ***P*(*k* + 1)** is true…

Proof Example

$$\sum_{i=1}^{k+1} i = 1 + 2 + 3 + ... + k + (k+1)$$

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This is just the sum from i=1 to k

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$$\sum_{i=1}^{k+1} i = 1 + 2 + 3 + ... + k + (k+1)$$

This is just the sum from i=1 to k

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$

We have assumed that ***P*(*k*)** is true

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$$

Proof Example

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Therefore if we assume ***P*(*k*)** is true, then ***P*(*k* + 1)** must also be true

Proof Example

Prove that if ***n*** is a positive integer, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Predicate: *P*(*n*): $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Goal: Show that ***P*(*n*)** is true for all ***n* ≥ 1**

<u>Base Case:</u> True!

<u>Inductive Case:</u> True!

Therefore since the Base Case and Inductive Case are both true, ***P*(*n*)** is true for all ***n* ≥ 1**

Quick Summary

Prove that for all ***n* ≥ *b***, "..." is true.

***P*(*n*):** "..."

<u>Base Case</u><u>:</u> Show that ***P*(*b*)** is true

<u>Inductive Case:</u> We must show that if ***P*(*k*)** is true, then ***P*(*k* + 1)** is true

    <u>Inductive Hypothesis:</u> Assume that ***P*(*k*)** is true for an arbitrary positive integer ***k* ≥ *b*.**

    We must now show that ***P*(*k* + 1):** "..." is true

    Given ***P*(*k* + 1) =** "..."

        More derivation…

    Given our Inductive Hypothesis …….. show that ***P*(*k* + 1)** must be true

    Therefore ***P*(*k*)** → ***P*(*k* + 1)**

Therefore both the Base Case and Inductive case are true, so ***P*(*n*):** "..." is true for all ***n* ≥ *b***

Outline

- Sequences
- Summations
- Recurrence Relations
- Mathematical Induction
- **Strong Induction**

Strong Induction

Problem: What happens when our next term (at ***k* + 1**) depends on multiple previous terms (more than just term ***k***)?

Strong Induction

Problem: What happens when our next term (at ***k*** **+ 1)** depends on multiple previous terms (more than just term ***k***)?

Example: The terms in the fibonacci sequence rely on two previous terms

Strong Induction

Principle of Strong Mathematical Induction

Let a, b be integers with $a \leq b$

Let $P(n)$ be a statement defined for any integer $n \geq a$

Then $P(n)$ is true for all $n \geq a$ if the following two conditions hold:

1. $P(a), P(a + 1), \ldots, P(b)$ are all individually true (the base cases)
2. For all $k \geq b$, $P(a) \wedge P(a + 1) \wedge \ldots \wedge P(k) \rightarrow P(k + 1)$ (the inductive case)

Proof Template

To formally prove something via strong induction, you must do **all** of the following:

1. Express the statement being proved in the form " $\forall n \geq a, P(n)$ for a fixed integer a
2. Prove the **Base Cases**: show that $P(a), P(a + 1), \ldots, P(b)$ are all true.
3. Prove the **Inductive Case**
   a. Inductive Hypothesis: Assume $P(i)$ is true for all i, where $a \leq i \leq k$ for an arbitrary $k \geq b$
   b. State what must be proved under this assumption; write out $P(k + 1)$
   c. Prove the statement $P(k + 1)$ is true by using the inductive assumption
   d. Clearly identify the conclusion
4. Now that you have proven the Base Case and Inductive Case, conclude that $P(n)$ is true for all $n \geq a$ by the principle of strong mathematical induction

Strong Mathematical Induction Example

Prove that a class of $n \geq 12$ students can be divided into groups of 4 or 5

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***P*(*n*):** For all $n \geq 12$, a class of size n can be divided into groups of 4 or 5

Base Cases:

Prove $P(n)$ is true for $n = 12, 13, 14,$ and 15

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***P*(*n*):** For all $n \geq 12$, a class of size n can be divided into groups of 4 or 5

Base Cases:

Prove $P(n)$ is true for $n = 12, 13, 14$, and 15

$12 = 4 + 4 + 4$
$13 = 4 + 4 + 5$
$14 = 4 + 5 + 5$
$15 = 5 + 5 + 5$

Strong Mathematical Induction Example

Prove that a class of ***n*** ≥ 12 students can be divided into groups of 4 or 5

P*(*n*):** For all ***n ≥ 12, a class of size ***n*** can be divided into groups of 4 or 5

<u>Base Cases:</u>

Prove ***P*(*n*)** is true for ***n*** = 12, 13, 14, and 15

12 = 4 + 4 + 4
13 = 4 + 4 + 5
14 = 4 + 5 + 5
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Therefore *P*(12), *P*(13), *P*(14), and *P*(15) are all TRUE

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a *b*

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Strong Mathematical Induction Example

Prove that a class of $n \geq 12$ students can be divided into groups of 4 or 5

$P(n)$: For all $n \geq 12$, a class of size n can be divided into groups of 4 or 5

Inductive Case:

We must show that if $P(i)$ is true for $12 \leq i \leq k$, where $k \geq 15$, then $P(k + 1)$ must be true.

Strong Mathematical Induction Example

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***P*(*n*):** For all $n \geq 12$, a class of size n can be divided into groups of 4 or 5

Inductive Case:

We must show that if $P(i)$ is true for $12 \leq i \leq k$, where $k \geq 15$, then $P(k + 1)$ must be true.

Inductive Hypothesis: Let $k \geq 15$

Assume $P(i)$ is true for all i, where $12 \leq i \leq k$. Notably, $P(k - 3)$ is true…

Strong Mathematical Induction Example

Proof of the Inductive Case:

We must show that $P(k + 1)$ is true.

From the $k + 1$ students, form a group of 4 students. Now $(k + 1) - 4 = k - 3$ students remain to be grouped.

By our inductive hypothesis $P(k - 3)$ is true, so we can divide the remaining $k - 3$ students into groups of 4 or 5.

Therefore if $P(i)$ is true for all i, where $12 \leq i \leq k$, then $P(k + 1)$ must be true

Strong Mathematical Induction Example

Prove that a class of ***n*** ≥ 12 students can be divided into groups of 4 or 5

P*(*n*):** For all ***n ≥ 12, a class of size ***n*** can be divided into groups of 4 or 5

Base Cases: TRUE

Inductive Case: TRUE

Therefore by the principle of strong mathematical induction, we can conclude that ***P*(*n*)** is true for all ***n*** ≥ 12

One Last Induction Example

Let $\{a_n\}$ be the sequence defined by $a_1 = 1$, $a_2 = 8$, $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$.

Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$

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Do we need strong mathematical induction here?

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Do we need strong mathematical induction here? **Yes.** The later terms in our sequence rely on more than just the previous term.

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Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$

1. Express the statement being proved in the form " $\forall n \geq a, P(n)$ for a fixed integer a
2. Prove the **Base Cases**: show that $P(a), P(a + 1), \ldots, P(b)$ are all true.
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   a. Inductive Hypothesis: Assume $P(i)$ is true for all i, where $a \leq i \leq k$ for an arbitrary $k \geq b$
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4. Now that you have proven the Base Case and Inductive Case, conclude that $P(n)$ is true for all $n \geq a$ by the principle of strong mathematical induction

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Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$

1. Express the statement being proved in the form " $\forall n \geq a$, $P(n)$ for a fixed integer a

$P(n)$: $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$, we will now show that for all $n \in \mathbb{N}$, $P(n)$ is true

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Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$

2. Prove the **Base Cases**: show that $P(a), P(a + 1), \ldots, P(b)$ are all true.

$P(1)$: $a_1 = 1$, $a_1 = 3 \cdot 2^{n-1} + 2(-1)^n = 3 \cdot 2^0 + 2(-1)^1 = 3 - 2 = 1$ ✓

$P(2)$: $a_2 = 8$, $a_2 = 3 \cdot 2^{n-1} + 2(-1)^n = 3 \cdot 2^1 + 2(-1)^2 = 6 + 2 = 8$ ✓

Therefore our base cases, $P(1)$ and $P(2)$ are both true

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Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$

3. Prove the **Inductive Case**
   a. **Inductive Hypothesis:** Assume $P(i)$ is true for all i, where $a \leq i \leq k$ for an arbitrary $k \geq b$

Assume $P(i)$: $a_i = 3 \cdot 2^{i-1} + 2(-1)^i$ for all $1 \leq i \leq k$, where $k \geq 2$

One Last Induction Example

Let $\{a_n\}$ be the sequence defined by $a_1 = 1$, $a_2 = 8$, $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$.

Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$

1. Express the statement being proved in the form " $\forall n \geq a, P(n)$ for a fixed integer a
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Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$

b. State what must be proved under this assumption; write out ***P*(*k* + 1)**

Now we must prove that ***P*(k + 1)** is true. Specifically, we must show that

$a_{k+1} = 3 \cdot 2^{k+1-1} + 2(-1)^{k+1}$

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c. Prove the statement **$P(k + 1)$** is true by using the inductive assumption

We must show $a_{k+1} = 3 \cdot 2^{k+1-1} + 2(-1)^{k+1}$

$a_{k+1} = a_k + 2a_{k-1}$ (by definition of our recurrence relation)

$a_{k+1} = 3 \cdot 2^{k-1} + 2(-1)^k + 2(3 \cdot 2^{k-2} + 2(-1)^{k-2})$ (by subbing in our assumption)

$= 3\,(2^{k-1} + 2^{k-1}) = 2\,((-1)^k + 2(-1)^{k-1}) = 3 \cdot 2^k + 2(-1)^{k+1}$

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Let $\{a_n\}$ be the sequence defined by $a_1 = 1$, $a_2 = 8$, $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$.

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   c. Prove the statement $P(k + 1)$ is true by using the inductive assumption
   d. Clearly identify the conclusion
4. Now that you have proven the Base Case and Inductive Case, conclude that $P(n)$ is true for all $n \geq a$ by the principle of strong mathematical induction

One Last Induction Example

Let $\{a_n\}$ be the sequence defined by $a_1 = 1$, $a_2 = 8$, $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$.

Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$

d. Clearly identify the conclusion

Therefore $P(k+1)$ is true when $P(i)$ is true for all $1 \leq i \leq k$, where $k \geq 2$

So the proof of the induction step is complete

One Last Induction Example

Let $\{a_n\}$ be the sequence defined by $a_1 = 1$, $a_2 = 8$, $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$.

Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$

1. Express the statement being proved in the form " $\forall n \geq a, P(n)$ for a fixed integer a
2. Prove the **Base Cases**: show that $P(a), P(a+1), \ldots, P(b)$ are all true.
3. Prove the **Inductive Case**
   a. Inductive Hypothesis: Assume $P(i)$ is true for all i, where $a \leq i \leq k$ for an arbitrary $k \geq b$
   b. State what must be proved under this assumption; write out $P(k+1)$
   c. Prove the statement $P(k+1)$ is true by using the inductive assumption
   d. Clearly identify the conclusion
4. Now that you have proven the Base Case and Inductive Case, conclude that $P(n)$ is true for all $n \geq a$ by the principle of strong mathematical induction

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4. Now that you have proven the Base Case and Inductive Case, conclude that $P(n)$ is true for all $n \geq a$ by the principle of strong mathematical induction

By the principle of strong mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$