CSE 191

Introduction to Discrete Structures

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Introduction to Set Theory

Outline

- **Set Basics**
  - **Definition**
  - Universal Set
  - Cardinality
- Set Equality and Subsets
- Set Operations

Sets

A **set** is a **collection of objects** that do NOT have an order

Each object is called an **element** or **member** of the set

Notation:

- $\mathbf{e} \in \boldsymbol{S}$ means that **e** is an element of ***S***
- $\mathbf{e} \notin \boldsymbol{S}$ means that **e** is not an element of ***S***

Sets

How do we describe a set?

1. List all elements
   - ie: {1, 2, 3}
   - This is called **roster notation**
2. Provide a description of what the elements look like
   - ie: $\{ a \mid a > 2, a \in \mathbb{Z} \}$
   - This is called **set builder notation**

Common Sets

- $\mathbb{N} = \{1, 2, 3, \ldots\}$: the set of **natural numbers**
  - Sometimes 0 is considered a member, which some disagree with
- $\mathbb{Z} = \{0, -1, 1, -2, 2, \ldots\}$: the set of **integers**
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$: the set of **positive integers**
- $\mathbb{Q} = \{ p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$: the set of **rational numbers**
  - Numbers that can be written as a fraction of integers
- $\mathbb{Q}^+ = \{ x \mid x \in \mathbb{Q}, x > 0\}$: the set of **positive rational numbers**
- $\mathbb{R}$: the set of **real numbers**
- $\mathbb{R}^+ = \{x \mid x \in \mathbb{R}, x > 0\}$: the set of **positive real numbers**
- $\mathbb{C}$: the set of complex numbers

More Examples

A = {Red, Purple, Green, Brown} is a set containing 5 colors

B = {Monopoly, Scrabble, Catan} is a set containing 3 board games

C = {x | x takes CSE 191 in Spring 2023} is a set of ~300 students

D = {$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$} is a **set containing 4 sets**. It has 4 elements.

E = {x | x $\in \mathbb{Z}^+$, x < 5} is a set containing 4 integers, 1, 2, 3, and 4.

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Universal Set

When discussing sets, there is always a **universal set** ***U*** involved, which contains all objects under consideration

- ie: for A = {Red, Purple, Green, Brown}, the universal set might be all colors
- ie: for B = {Monopoly, Scrabble, Catan} the universal set might be all board games

In most cases, the universal set is **implicit and omitted from discussion**

Universal Universal Set

Is there a universal set covering all universes? (Russell's Paradox)

Consider a book named ***Book Titles*** that contains a list of the titles of every book that does not contain its own title.

Does ***Book Titles*** contain a line for ***Book Titles?***

Universal Universal Set

Is there a universal set covering all universes? (Russell's Paradox)

Consider a book named ***Book Titles*** that contains a list of the titles of every book that does not contain its own title.

Does ***Book Titles*** contain a line for ***Book Titles?***

Yes? Now ***Book Titles*** is a book containing its own title, so it shouldn't be listed

No? Now ***Book Titles*** is a book not containing its own title, so it should be listed

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- **Set Basics**
  - Definition
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Cardinality (for Finite Sets)

If a set ***A*** contains exactly ***n*** elements, where ***n*** is a non-negative integer, then ***A*** is a **finite set**.

n is called the **cardinality** of ***A***, denoted by |***A***|.

The **empty set** or **null set** is the set that contains no elements, denoted by ∅ or {}. It has size 0.

Cardinality (for Finite Sets)

Do we count duplicate items?

NO. We only count unique items for cardinality.

Consider the following sets:

- F = {Apple, Banana, Apple, Orange, Orange, Apple}
- F' = {Apple, Banana, Orange}

F = F' and |F| = |F'| = 3.

Cardinality Examples

1. $|\{x \mid -2 < x < 5, x \in \mathbb{Z}\}| = 6$, the elements are -1, 0, 1, 2, 3, 4
2. $|\varnothing| = 0$, no elements in the empty set
3. $|\{x \mid x \in \varnothing, x < 3\}| = 0$, because no x satisfies $x \in \varnothing$
4. $|\{x \mid x \in \{\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}\}\}| = 4$, the elements are $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$
5. $|\{0,0,0,1,1,1,2,2,2,3,3,,4\}| = 5$, the elements are 0,1,2,3,4

Cardinality: ∅ vs {∅}

Consider the contents of this shopping cart →

|shopping cart| = 0, shopping cart = ∅ or {}

Cardinality: ∅ vs {∅}

Consider the contents of this shopping basket →

|shopping basket| = 0, shopping basket = ∅ or {}

Cardinality: ∅ vs {∅}

Now, consider the contents of this shopping cart →

|shopping card| = 1 (it contains the basket)

The set representation of the shopping cart is {∅}

It is a set containing one item, the empty set.

Cardinality (for Infinite Sets)

If ***A*** is not finite, then it is an **infinite set**

What is the cardinality (the size) of an infinite set?

Do all infinite sets have the same size?

- Are there more rational numbers than integers?
- Are there more real numbers than rational numbers?

Cardinality (for Infinite Sets)

If ***A*** is not finite, then it is an **infinite set**

What is the cardinality (the size) of an infinite set?

Do all infinite sets have the same size? **No**

- Are there more rational numbers than integers?
- Are there more real numbers than rational numbers?
- Only one of the above is "yes"

Outline

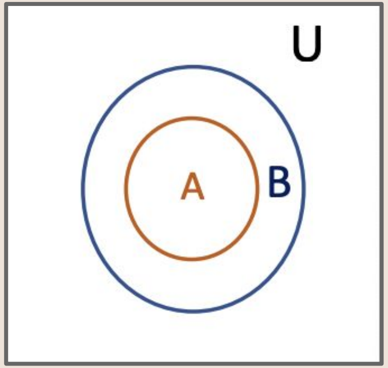
- Set Basics
- **Set Equality and Subsets**
    - **Subsets**
    - Set Equality
- Set Operations

Subsets

A set $\boldsymbol{A}$ is a **subset** of $\boldsymbol{B}$ if and only if every element of $\boldsymbol{A}$ is also in $\boldsymbol{B}$.
Denoted by $\boldsymbol{A} \subseteq \boldsymbol{B}$
If $\boldsymbol{A} \subseteq \boldsymbol{B}$, then $\forall x \in A, x \in B$

Note: for any set $\boldsymbol{A}$, $\varnothing \subseteq \boldsymbol{A}$ and $\boldsymbol{A} \subseteq \boldsymbol{A}$

If $\boldsymbol{A} \subseteq \boldsymbol{B}$ but $\boldsymbol{A} \neq \boldsymbol{B}$, then $\boldsymbol{A}$ is a **proper subset** of $\boldsymbol{B}$.
Denoted by $\boldsymbol{A} \subset \boldsymbol{B}$ or $\boldsymbol{A} \subsetneq \boldsymbol{B}$



Venn Diagram showing A ⊆ B

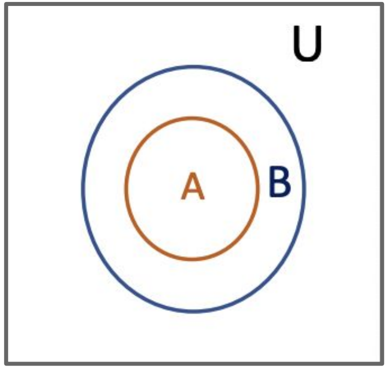
Subsets

To show that *A* is a subset of *B*:

- Prove that if $x \in A$ then $x \in B$

To show that *A* is not a subset of *B*:

- Find a counterexample; an x s.t. $x \in A$ but $x \notin B$



Venn Diagram showing $A \subseteq B$

Subset Examples

1. $\{1, 2\} \subseteq \{2, 1, 3\}$
   - Also, $\{1, 2\} \subset \{2, 1, 3\}$
2. $\{x \in \mathbb{Z} \mid x \text{ is even}\} \subseteq \{x \mid x \in \mathbb{Z}\}$
   - Every even integer is an integer
3. $\{x \in \mathbb{Z} \mid x \text{ is even}\} \nsubseteq \{x \mid x \in \mathbb{Z} \text{ and } 1 < x < 5\}$
   - Both sets share 2 and 4, but 3 is not in the second set
4. $\{2, 4, 6, 8, \ldots\} \subseteq \{n \in \mathbb{N} \mid n \text{ is even}\}$
   - $\subseteq$ is still true even when the sets are the same

More Examples

Let $\boldsymbol{A} = \{a, b, c\}$

$\boldsymbol{B} = \{a, b, e\}$

$\boldsymbol{C} = \{a, e\}$

Which of the following are true?

1. $\boldsymbol{A} \subseteq \boldsymbol{B}$
2. $\boldsymbol{A} \subset \boldsymbol{B}$
3. $\boldsymbol{C} \subset \boldsymbol{B}$
4. $\boldsymbol{C} \subseteq \boldsymbol{C}$
5. $\{b\} \subseteq \{a, \{b\}, c\}$
6. $\varnothing \subseteq \{a, \{b\}, c\}$

More Examples

Let $\boldsymbol{A} = \{a, b, c\}$

$\boldsymbol{B} = \{a, b, e\}$

$\boldsymbol{C} = \{a, e\}$

Which of the following are true?

1. $\boldsymbol{A} \subseteq \boldsymbol{B}$ FALSE
2. $\boldsymbol{A} \subset \boldsymbol{B}$ FALSE
3. $\boldsymbol{C} \subset \boldsymbol{B}$ TRUE
4. $\boldsymbol{C} \subseteq \boldsymbol{C}$ TRUE
5. $\{b\} \subseteq \{a, \{b\}, c\}$ FALSE
6. $\varnothing \subseteq \{a, \{b\}, c\}$ TRUE

Distinction between ∈ and ⊆

Remember: $\boldsymbol{x} \in \boldsymbol{S}$ means $\boldsymbol{x}$ is an element of $\boldsymbol{S}$. $\boldsymbol{S_1} \subseteq \boldsymbol{S_2}$ means $\boldsymbol{S_1}$ is a subset of $\boldsymbol{S_2}$

Which of the following are true?

1. $b \in \{a, \{b\}, c\}$
2. $\{b\} \in \{a, \{b\}, c\}$
3. $\{b\} \subseteq \{a, \{b\}, c\}$
4. $\{b\} \subseteq \{a, \{b\}, b, c\}$

Distinction between ∈ and ⊆

Remember: $\boldsymbol{x} \in \boldsymbol{S}$ means $\boldsymbol{x}$ is an element of $\boldsymbol{S}$. $\boldsymbol{S_1} \subseteq \boldsymbol{S_2}$ means $\boldsymbol{S_1}$ is a subset of $\boldsymbol{S_2}$

Which of the following are true?

1. $b \in \{a, \{b\}, c\}$ FALSE (b is not in the set)
2. $\{b\} \in \{a, \{b\}, c\}$ TRUE ({b} is in the set)
3. $\{b\} \subseteq \{a, \{b\}, c\}$ FALSE (b is an element in the left set but not the right)
4. $\{b\} \subseteq \{a, \{b\}, b, c\}$ TRUE (b is now an element in the left and right set)

Outline

- Set Basics
- **Set Equality and Subsets**
  - Subsets
  - **Set Equality**
- Set Operations

Set Equality

Fact: Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are sets. Then $\boldsymbol{A} = \boldsymbol{B}$ if and only if $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $\boldsymbol{B} \subseteq \boldsymbol{A}$

<u>To Prove Set Equality</u>

Set Equality

Fact: Suppose ***A*** and ***B*** are sets. Then ***A* = *B*** if and only if ***A*** ⊆ ***B*** and ***B*** ⊆ ***A***

<u>To Prove Set Equality</u>

<u>Prove *A* ⊆ *B*:</u>

Assume ***x*** in ***A***

…

∴ ***x*** in ***B*** as well

Conclude that ***A*** ⊆ ***B***

Set Equality

Fact: Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are sets. Then $\boldsymbol{A} = \boldsymbol{B}$ if and only if $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $\boldsymbol{B} \subseteq \boldsymbol{A}$

To Prove Set Equality

Prove $\boldsymbol{A} \subseteq \boldsymbol{B}$:

Assume $\boldsymbol{x}$ in $\boldsymbol{A}$

…

$\therefore$ $\boldsymbol{x}$ in $\boldsymbol{B}$ as well

Conclude that $\boldsymbol{A} \subseteq \boldsymbol{B}$

Prove $\boldsymbol{B} \subseteq \boldsymbol{A}$:

Assume $\boldsymbol{y}$ in $\boldsymbol{B}$

…

$\therefore$ $\boldsymbol{y}$ in $\boldsymbol{A}$ as well

Conclude that $\boldsymbol{B} \subseteq \boldsymbol{A}$

Set Equality

Fact: Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are sets. Then $\boldsymbol{A} = \boldsymbol{B}$ if and only if $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $\boldsymbol{B} \subseteq \boldsymbol{A}$

To Prove Set Equality

Prove $\boldsymbol{A} \subseteq \boldsymbol{B}$:

Assume $\boldsymbol{x}$ in $\boldsymbol{A}$

...

$\therefore$ $\boldsymbol{x}$ in $\boldsymbol{B}$ as well

Conclude that $\boldsymbol{A} \subseteq \boldsymbol{B}$

Prove $\boldsymbol{B} \subseteq \boldsymbol{A}$:

Assume $\boldsymbol{y}$ in $\boldsymbol{B}$

...

$\therefore$ $\boldsymbol{y}$ in $\boldsymbol{A}$ as well

Conclude that $\boldsymbol{B} \subseteq \boldsymbol{A}$

Conclude that since $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $\boldsymbol{B} \subseteq \boldsymbol{A}$ then $\boldsymbol{A} = \boldsymbol{B}$

Equality via Subsets

Let ***A*** = {1, 2, 3, 4} and ***B*** = {x | ***x*** ∈ ℤ and 1 ≤ ***x*** < 5}. Prove ***A* = *B*.**

Equality via Subsets

Let $\boldsymbol{A} = \{1, 2, 3, 4\}$ and $\boldsymbol{B} = \{x \mid \boldsymbol{x} \in \mathbb{Z} \text{ and } 1 \leq \boldsymbol{x} < 5\}$. Prove $\boldsymbol{A} = \boldsymbol{B}$.

Proof of $\boldsymbol{A} \subseteq \boldsymbol{B}$:

Assume $\boldsymbol{x} \in \boldsymbol{A}$

Case x = 1: $1 \in \mathbb{Z}$ and $1 \leq 1 < 5$

$\therefore 1 \in \boldsymbol{B}$

Case x = 2: $2 \in \mathbb{Z}$ and $1 \leq 2 < 5$

$\therefore 2 \in \boldsymbol{B}$

Case x = 3: $3 \in \mathbb{Z}$ and $1 \leq 3 < 5$

$\therefore 3 \in \boldsymbol{B}$

Case x = 4: $4 \in \mathbb{Z}$ and $1 \leq 4 < 5$

$\therefore 4 \in \boldsymbol{B}$

$\therefore x \in \boldsymbol{B}$, so $\boldsymbol{A} \subseteq \boldsymbol{B}$

Equality via Subsets

Let $\boldsymbol{A} = \{1, 2, 3, 4\}$ and $\boldsymbol{B} = \{x \mid \boldsymbol{x} \in \mathbb{Z} \text{ and } 1 \leq \boldsymbol{x} < 5\}$. Prove $\boldsymbol{A} = \boldsymbol{B}$.

Proof of $A \subseteq B$:

Assume $\boldsymbol{x} \in \boldsymbol{A}$

Case x = 1: $1 \in \mathbb{Z}$ and $1 \leq 1 < 5$

$\therefore 1 \in \boldsymbol{B}$

Case x = 2: $2 \in \mathbb{Z}$ and $1 \leq 2 < 5$

$\therefore 2 \in \boldsymbol{B}$

Case x = 3: $3 \in \mathbb{Z}$ and $1 \leq 3 < 5$

$\therefore 3 \in \boldsymbol{B}$

Case x = 4: $4 \in \mathbb{Z}$ and $1 \leq 4 < 5$

$\therefore 4 \in \boldsymbol{B}$

$\therefore x \in \boldsymbol{B}$, so $\boldsymbol{A} \subseteq \boldsymbol{B}$

Proof of $B \subseteq A$:

Assume $\boldsymbol{x} \in \boldsymbol{B}$

$\boldsymbol{x} \in \mathbb{Z}$ and $1 \leq \boldsymbol{x} < 5$

So $\boldsymbol{x}$ must be 1, 2, 3, or 4

If $\boldsymbol{x}$ is 1, 2, 3, or 4, then $\boldsymbol{x} \in \boldsymbol{A}$

$\therefore x \in \boldsymbol{A}$, so $\boldsymbol{B} \subseteq \boldsymbol{A}$

Equality via Subsets

Let $A = \{1, 2, 3, 4\}$ and $B = \{x \mid x \in \mathbb{Z} \text{ and } 1 \leq x < 5\}$. Prove $A = B$.

Proof of $A \subseteq B$:

Assume $x \in A$

Case $x = 1$: $1 \in \mathbb{Z}$ and $1 \leq 1 < 5$

$\therefore 1 \in B$

Case $x = 2$: $2 \in \mathbb{Z}$ and $1 \leq 2 < 5$

$\therefore 2 \in B$

Case $x = 3$: $3 \in \mathbb{Z}$ and $1 \leq 3 < 5$

$\therefore 3 \in B$

Case $x = 4$: $4 \in \mathbb{Z}$ and $1 \leq 4 < 5$

$\therefore 4 \in B$

$\therefore x \in B$, so $A \subseteq B$

Proof of $B \subseteq A$:

Assume $x \in B$

$x \in \mathbb{Z}$ and $1 \leq x < 5$

So x must be 1, 2, 3, or 4

If x is 1, 2, 3, or 4, then $x \in A$

$\therefore x \in A$, so $B \subseteq A$

Since $A \subseteq B$ and $B \subseteq A$ we get that $A = B$

Equality via Subsets

Let E_1 = {{}} and E_2 = {∅, {}}. Prove $E_1 = E_2$.

Equality via Subsets

Let $\boldsymbol{E_1} = \{\{\}\}$ and $\boldsymbol{E_2} = \{\varnothing, \{\}\}$. Prove $\boldsymbol{E_1} = \boldsymbol{E_2}$.

Proof of $E_1 \subseteq E_2$:

Assume $\boldsymbol{x} \in \boldsymbol{E_1}$

Case x = {}: $\{\} \in \boldsymbol{E_2}$

This is the only element in $\boldsymbol{E_1}$

$\therefore \boldsymbol{E_1} \subseteq \boldsymbol{E_2}$

Equality via Subsets

Let $E_1 = \{\{\}\}$ and $E_2 = \{\varnothing, \{\}\}$. Prove $E_1 = E_2$.

Proof of $E_1 \subseteq E_2$:

Assume $x \in E_1$

Case $x = \{\}$: $\{\} \in E_2$

This is the only element in E_1

$\therefore E_1 \subseteq E_2$

Proof of $E_2 \subseteq E_1$:

Assume $x \in E_2$

Case $x = \{\}$: $\{\} \in E_1$

Case $x = \varnothing$: $\varnothing \in E_1$

$\therefore E_2 \subseteq E_1$

Equality via Subsets

Let $\boldsymbol{E_1}$ = {{}} and $\boldsymbol{E_2}$ = {∅, {}}. Prove $\boldsymbol{E_1} = \boldsymbol{E_2}$.

Proof of $E_1 \subseteq E_2$:

Assume $\boldsymbol{x} \in \boldsymbol{E_1}$

Case x = {}: {} $\in \boldsymbol{E_2}$

This is the only element in $\boldsymbol{E_1}$

$\therefore \boldsymbol{E_1} \subseteq \boldsymbol{E_2}$

Proof of $E_2 \subseteq E_1$:

Assume $\boldsymbol{x} \in \boldsymbol{E_2}$

Case x = {}: {} $\in \boldsymbol{E_1}$

Case x = ∅: ∅ $\in \boldsymbol{E_1}$

$\therefore \boldsymbol{E_2} \subseteq \boldsymbol{E_1}$

Since $\boldsymbol{E_1} \subseteq \boldsymbol{E_2}$ and $\boldsymbol{E_2} \subseteq \boldsymbol{E_1}$ we get that $\boldsymbol{E_1} = \boldsymbol{E_2}$, notice duplicates don't

Outline

- Set Basics
- Set Equality and Subsets
- **Set Operations**
  - **Basic Operators**
  - Power Set
  - Cartesian Product
  - Partitions

Set Operations

We have: +, -, ×, ÷, … operators for numbers

We have: ∨, ∧, ¬, →, … operators for propositions

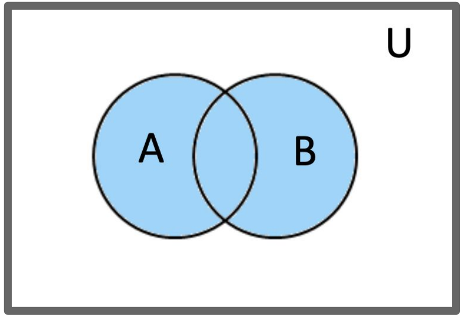
Set Operation	Symbol	Idea	Logic
Union of ***A*** and ***B***	***A*** ∪ ***B***	in ***A*** or ***B***	∨
Intersection of ***A*** and ***B***	***A*** ∩ ***B***	in ***A*** and ***B***	∧
Complement of ***A***	$\bar{A}$	not in ***A***	¬
Difference of ***A*** and ***B***	***A*** \ ***B***	in ***A*** and not in ***B***	***A*** ∧ ¬***B***
Symmetric difference of ***A*** and ***B***	***A*** ○ ***B***	in ***A*** or ***B***, not both	⊕
A is subset of ***B***	***A*** ⊆ ***B***	if in ***A*** then in ***B***	→

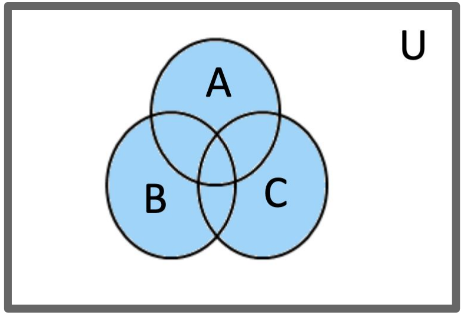
Set Union

The **union** of two sets, ***A*** and ***B***, is the set that contains exactly all elements that are in ***A*** or ***B*** (or in both)

- Denoted by ***A*** ∪ ***B***
- Formally, ***A*** ∪ ***B*** = { x | x ∈ ***A*** or x ∈ ***B*** }

A ∪ ***B*** is shaded →





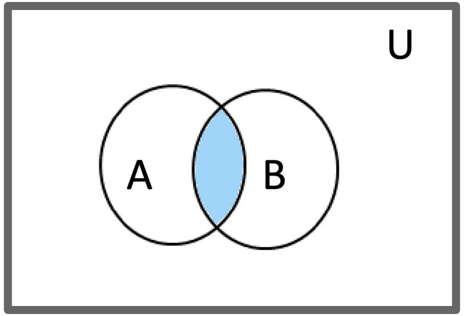
← ***A*** ∪ ***B*** ∪ ***C*** is shaded

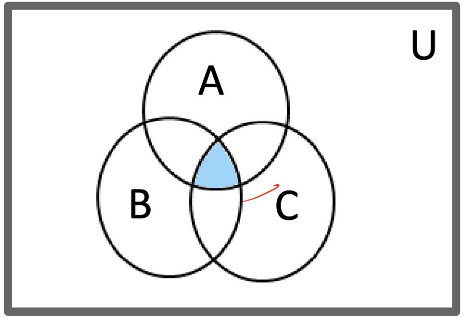
Set Intersection

The **intersection** of two sets, ***A*** and ***B***, is the set that contains exactly all elements that are in ***A*** and ***B***

- Denoted by $A \cap B$
- Formally, $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

$A \cap B$ is shaded →





← $A \cap B \cap C$ is shaded

Set Intersection

The **intersection** of two sets, ***A*** and ***B***, is the set that contains exactly all elements that are in ***A*** and ***B***

- Denoted by $\boldsymbol{A} \cap \boldsymbol{B}$
- Formally, $\boldsymbol{A} \cap \boldsymbol{B} = \{ x \mid x \in \boldsymbol{A} \text{ and } x \in \boldsymbol{B} \}$

Two sets are **disjoint** if their intersection is the empty set

Principle of Inclusion-Exclusion

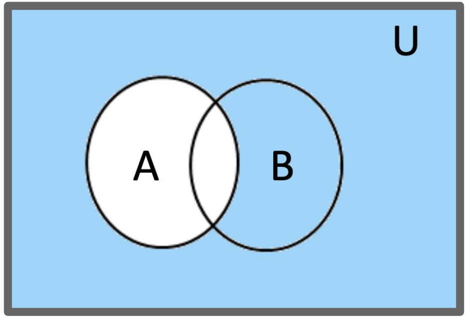
$|\boldsymbol{A} \cup \boldsymbol{B}| = |\boldsymbol{A}| + |\boldsymbol{B}| - |\boldsymbol{A} \cap \boldsymbol{B}|$

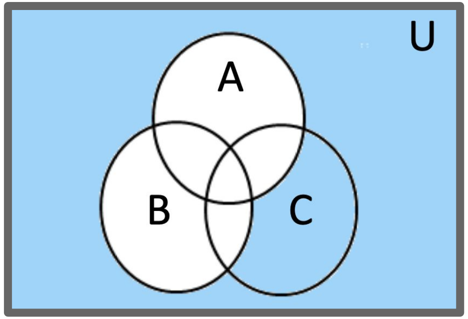
Set Complement

The **complement** of set ***A*** is the set that contains exactly all the elements that are not in ***A***.

- Denoted by $\overline{A}$
- Formally, $\overline{A} = \{ x \mid x \notin A \}$

$\overline{A}$ is shaded →





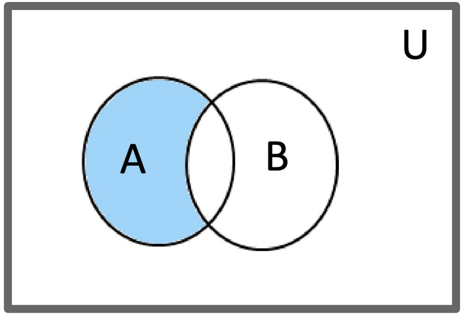
← $\overline{A \cup B}$ is shaded

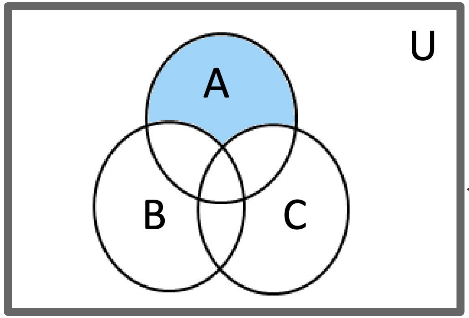
Set Difference

The **difference** of set ***A*** and set ***B*** is the set that contains exactly all elements that are in ***A*** but not in ***B***

- Denoted by $A - B$ (or $A \setminus B$)
- Formally, $A - B = \{ x \mid x \in A \text{ and } x \notin B \} = A \cap \overline{B}$

$A - B$ is shaded →





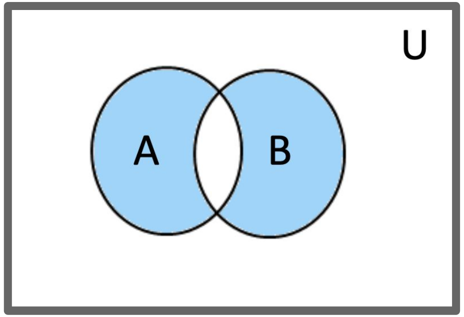
← $A - (B \cup C)$ is shaded

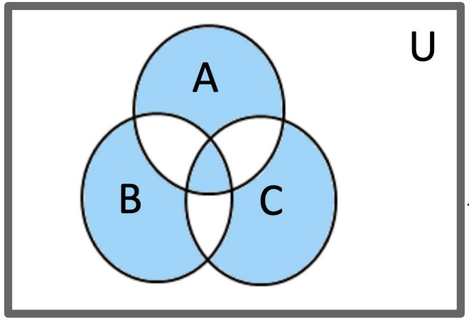
Symmetric Difference

The **symmetric difference** of set ***A*** and set ***B*** is the set that contains all elements that are in exactly one of ***A*** or ***B***

- Denoted by $\boldsymbol{A \oplus B}$ (or ***A*** ○ ***B***)
- Formally, $\boldsymbol{A \oplus B = (A - B) \cup (B - A)}$

$\boldsymbol{A \oplus B}$ is shaded →





← $\boldsymbol{A \oplus B \oplus C}$ is shaded

It includes values that are in an odd number of sets, ie $\{ x \mid x \in \boldsymbol{A} \oplus x \in \boldsymbol{B} \oplus x \in \boldsymbol{C} \}$

Examples

Let the universe be $\mathbb{Z}^+$. Write the contents of ***A*** in roster form, where:

$$A = (\{\ x \mid x \text{ is even}\} - \{\ x \mid x \text{ is a multiple of } 3\}) \cap \{\ y \mid y \leq 10\ \}$$

Examples

Let the universe be $\mathbb{Z}^+$. Write the contents of ***A*** in roster form, where:

A = ({ x | x is even} - { x | x is a multiple of 3}) ∩ { y | y ≤ 10 }

{ 2, 4, 8, 10 }

Examples

Let the universe be the 7 colors in a rainbow (Red, Orange, Yellow, Green, Blue, Indigo, Violet). Write the contents of ***C*** and ***D*** in roster form, where:

$$C = (\{ c \mid c \text{ is 6 letters} \} \cup \{ c \mid c \text{ has odd length} \}) \oplus \{\text{Red, Blue, Yellow}\}$$

$$D = \overline{C}$$

Examples

Let the universe be the 7 colors in a rainbow (Red, Orange, Yellow, Green, Blue, Indigo, Violet). Write the contents of ***C*** and ***D*** in roster form, where:

C = ({ c | c is 6 letters } ∪ { c | c has odd length }) ⊕ {Red, Blue, Yellow}

$D = \overline{C}$

C = { Orange, Green, Indigo, Violet, Blue }

D = { Red, Yellow }

More Practice

Consider the universe { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Let $\boldsymbol{A}$ = { 1, 2, 3, 4, 5} and $\boldsymbol{B}$ = { 1, 2, 3, 4, 5, 6, 7, 8 }

1. $\boldsymbol{A} \cap \boldsymbol{B}$
2. $\boldsymbol{A} \cup \boldsymbol{B}$
3. $\overline{\boldsymbol{A}}$
4. $\overline{\boldsymbol{B}}$
5. $\boldsymbol{A} - \boldsymbol{B}$
6. $\boldsymbol{B} - \boldsymbol{A}$
7. $\boldsymbol{A} \oplus \boldsymbol{B}$

More Practice

Consider the universe { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Let $\boldsymbol{A}$ = { 1, 2, 3, 4, 5} and $\boldsymbol{B}$ = { 1, 2, 3, 4, 5, 6, 7, 8 }

1. $\boldsymbol{A} \cap \boldsymbol{B}$ = { 1, 2, 3, 4, 5 }
2. $\boldsymbol{A} \cup \boldsymbol{B}$ = { 1, 2, 3, 4, 5, 6, 7, 8 }
3. $\overline{\boldsymbol{A}}$ = { 6, 7, 8, 9, 10 }
4. $\overline{\boldsymbol{B}}$ = { 9, 10 }
5. $\boldsymbol{A} - \boldsymbol{B}$ = $\varnothing$
6. $\boldsymbol{B} - \boldsymbol{A}$ = { 6, 7, 8 }
7. $\boldsymbol{A} \oplus \boldsymbol{B}$ = $(\boldsymbol{A} - \boldsymbol{B}) \cup (\boldsymbol{B} - \boldsymbol{A})$ = { 6, 7, 8 }

Generalized Set Operators

We can simplify the notation for operating on ***n*** sets

For unions: $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$

Formally: $\bigcup_{i=1}^{n} A_i = \{s \mid s \in A_1 \text{ or } s \in A_2 \text{ or } \ldots \text{ or } A_n\}$

For Intersection: $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$

Formally: $\bigcap_{i=1}^{n} A_i = \{s \mid s \in A_1 \text{ and } s \in A_2 \text{ and } \ldots \text{ and } A_n\}$

Examples

Let $\boldsymbol{A_i}$ **= { 1, 2, 3, ..., *i* }** for all positive integers ***i***. Then compute:

$$\bigcup_{i=1}^{n} A_i \qquad\qquad \bigcap_{i=1}^{n} A_i$$

Examples

Let A_i = { 1, 2, 3, ..., i } for all positive integers i. Then compute:

$$\bigcup_{i=1}^{n} A_i = A_n \qquad \bigcap_{i=1}^{n} A_i = A_1$$

Examples

Let $\boldsymbol{B_i}$ **= { *i* + 1, *i* + 2, ..., 2*i* }** and $\boldsymbol{C_i}$ **= { *i* }**. Compute the following:

$$\bigcup_{i=1}^{n} B_i \qquad \bigcap_{i=1}^{n} B_i$$

$$\bigcup_{i=1}^{n} C_i \qquad \bigcap_{i=1}^{n} C_i$$

Examples

Let B_i = { i + 1, i + 2, ..., 2i } and C_i = { i }. Compute the following:

$$\bigcup_{i=1}^{n} B_i = A_{2n} - \{ 1 \}$$

$$\bigcap_{i=1}^{n} B_i = \varnothing$$

$$\bigcup_{i=1}^{n} C_i = A_n$$

$$\bigcap_{i=1}^{n} C_i = \varnothing$$

Prove that: $\bigcup_{i=1}^{n} C_i = \bigcup_{i=1}^{n} A_i$

Outline

- Set Basics
- Set Equality and Subsets
- **Set Operations**
  - Basic Operators
  - **Power Set**
  - Cartesian Product
  - Partitions

Power Set

The **power set** of set ***A*** is the set of all possible subsets of ***A***

Denoted by $\mathscr{P}(A)$

In general, $|\mathscr{P}(A)| = 2^{|A|}$

For any set ***A***, it is always the case that:

- $\varnothing \in \mathscr{P}(A)$ (the empty set is a subset of ***A*** …and every other set)
- $A \in \mathscr{P}(A)$ (***A*** is a subset of itself…every elements of ***A*** is in ***A***)

Power Set Example

$$\mathscr{P}(\{0, 1, 2\}) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

Note: For each subset, each element of our original set is either in it, or not in it. That's two options for each element, which is why there are 2^n possible subsets.

Power Set Exercises

Let $\boldsymbol{A} = \{a, \{a\}, \{a,b\}, b, \{c\}, d\}$

1. $a \in \boldsymbol{A}$
2. $\{b\} \subseteq \boldsymbol{A}$
3. $c \in \boldsymbol{A}$
4. $\{a, d\} \in \boldsymbol{A}$
5. $\{a, b\} \in \boldsymbol{A}$
6. $\{a, d\} \subseteq \boldsymbol{A}$
7. $\{a, d\} \in \mathscr{P}(\boldsymbol{A})$
8. $\mathscr{P}(\varnothing)$
9. $\mathscr{P}(\{\varnothing\})$
10. $\mathscr{P}(\{a\})$

Power Set Exercises

Let $\boldsymbol{A}$ = {a, {a}, {a,b}, b, {c}, d}

1. a $\in$ $\boldsymbol{A}$ **TRUE**
2. {b} $\subseteq$ $\boldsymbol{A}$ **TRUE**
3. c $\in$ $\boldsymbol{A}$ **FALSE**
4. {a, d} $\in$ $\boldsymbol{A}$ **FALSE**
5. {a, b} $\in$ $\boldsymbol{A}$ **TRUE**
6. {a, d} $\subseteq$ $\boldsymbol{A}$ **TRUE**
7. {a, d} $\in$ $\mathscr{P}(\boldsymbol{A})$ **TRUE**
8. $\mathscr{P}(\varnothing)$ **= { ∅ }**
9. $\mathscr{P}(\{\varnothing\})$ **= { ∅, {∅} }**
10. $\mathscr{P}(\{a\})$ **= { ∅, {a} }**

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Imposing Order on Elements

Sometimes order **is** important…

How can we impose order on elements?

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How can we impose order on elements?

An **ordered *n* tuple** ($\boldsymbol{a_1}$, $\boldsymbol{a_2}$, …, $\boldsymbol{a_n}$) has $\boldsymbol{a_1}$ as its first element, $\boldsymbol{a_2}$ as its second, …, and $\boldsymbol{a_n}$ as its $\boldsymbol{n^{th}}$ element.

Imposing Order on Elements

Sometimes order **is** important…

How can we impose order on elements?

An **ordered *n* tuple** ($a_1, a_2, \ldots, a_n$) has a_1 as its first element, a_2 as its second, …, and a_n as its n^{th} element.

Order is important for tuples. Assume $a_1 \neq a_2$

- $(a_1, a_2) \neq (a_2, a_1)$ ← tuple comparison
- $\{a_1, a_2\} = \{a_2, a_1\}$ ← set comparison

Imposing Order: Cartesian Product

The **Cartesian product** of two sets $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ is defined as the set of ordered tuples $(\boldsymbol{a_1}, \boldsymbol{a_2})$ where $\boldsymbol{a_1} \in \boldsymbol{A_1}$ and $\boldsymbol{a_2} \in \boldsymbol{A_2}$

- Denoted by $\boldsymbol{A_1} \times \boldsymbol{A_2}$
- Formally, $\boldsymbol{A_1} \times \boldsymbol{A_2} = \{(\boldsymbol{a_1}, \boldsymbol{a_2}) \mid \boldsymbol{a_1} \in \boldsymbol{A_1} \text{ and } \boldsymbol{a_2} \in \boldsymbol{A_2}\}$
- We say "$\boldsymbol{A_1}$ cross $\boldsymbol{A_2}$"

René Descartes

Computing the Cartesian Product

Example: $\{1, 2\} \times \{a, b, c\} = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

	a	b	c
1	(1,a)	(1,b)	(1,c)
2	(2,a)	(2,b)	(2,c)

$\{1, 2\} \times \{a, b, c\}$ is the set of all elements in our table

Computing the Cartesian Product

What if $\boldsymbol{A} = \{1, 2\}$ and $\boldsymbol{B} = \mathbb{Z}^+$...how do we compute $\boldsymbol{A} \times \boldsymbol{B}$

	1	2	3	...
1	**(1,1)**	**(1,2)**	**(1,3)**	**...**
2	**(2,1)**	**(2,2)**	**(2,3)**	**...**

$$\boldsymbol{A} \times \boldsymbol{B} = \{ (\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \in \{1, 2\}, \boldsymbol{y} \in \mathbb{Z}^+ \}$$

Notice how (1, 2) and (2, 1) are unique elements of $\boldsymbol{A} \times \boldsymbol{B}$

Generalized Cartesian Product

For $n \geq 2$, the cartesian product of A_1 to A_n is defined as follows:

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

Cartesian Power:

For any integer $n \geq 0$, $A^n = \begin{cases} \emptyset & n = 0 \\ A & n = 1 \\ A \times A \times \ldots \times A & n > 1 \end{cases}$

Formally, $A^n = \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in A\}$

Cartesian Product Examples

Let $\boldsymbol{A} = \{\, x \mid x \text{ is an odd integer in } \mathbb{Z}^{+} \text{ and } x < 6 \,\}$

$\boldsymbol{B} = \{\, y \mid y \text{ is an even integer in } \mathbb{Z}^{+} \text{ and } y < 8 \,\}$

$\boldsymbol{C} = \{\, 1, 2 \,\}$, $\boldsymbol{D} = \{\, 0, 1 \,\}$, $\boldsymbol{E} = \{\, a \,\}$

$\boldsymbol{A} \times \boldsymbol{B}$

$\boldsymbol{D}^3$

$\boldsymbol{C} \times \boldsymbol{C}$

$(\boldsymbol{C} \times \boldsymbol{E}) \times \boldsymbol{D}$

Cartesian Product Examples

Let $\boldsymbol{A} = \{ x \mid x \text{ is an odd integer in } \mathbb{Z}^+ \text{ and } x < 6 \}$

$\boldsymbol{B} = \{ y \mid y \text{ is an even integer in } \mathbb{Z}^+ \text{ and } y < 8 \}$

$\boldsymbol{C} = \{ 1, 2 \}$, $\boldsymbol{D} = \{ 0, 1 \}$, $\boldsymbol{E} = \{ a \}$

$\boldsymbol{A} \times \boldsymbol{B} = \{(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5, 2),(5,4),(5,6)\}$

$\boldsymbol{D}^3 = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$

$\boldsymbol{C} \times \boldsymbol{C} = \{(1,1), (1,2), (2,1), (2,2)\}$

$(\boldsymbol{C} \times \boldsymbol{E}) \times \boldsymbol{D} = \{((1,a),0),((1,a),1),((2,a),0),((2,a),1)$

Strings

How can we represent English words?

Strings

Let ***A* = {a, b, c, d, e, ..., z}** (the English alphabet)

- (c,a,t) and (d,o,g) are both members of $\boldsymbol{A^3}$
- (f,r,o,g) and (b,i,r,d) are both members of $\boldsymbol{A^4}$

We can shorthand tuples as words:

- cat, dog $\in \boldsymbol{A^3}$
- frog, bird $\in \boldsymbol{A^4}$

Strings

An **alphabet** is a *non-empty finite* set of symbols

A **string** is a finite sequence of symbols from an alphabet

- Shorthand for a tuple from the Cartesian power of an alphabet

The number of characters in a string is called th **length** of the string

- The length of string ***s*** is denoted by |***s***|

Example

The alphabet {0, 1} is used to form **binary strings**

$0001 \in \{0, 1\}^4$ $|0001| = 4$

$000 \in \{0, 1\}^3$ $|000| = 3$

$010111 \in \{0, 1\}^6$ $|010111| = 6$

$111 \in \{0, 1\}^3$ $|111| = 3$

Strings

What is the shortest string over any alphabet?

Strings

The smallest cartesian power is 0

- $\{a,b\}^0 = \{()\}$

How can we write the sequence of characters within ()?

- We let λ denote the **empty string**
- Then $\{a,b\}^0 = \{\lambda\}$
- $|\lambda| = 0$
- In programming, we usually denote the empty string with `""` or `''`

Strings

The smallest cartesian power is 0

- $\{a,b\}^0 = \{()\}$

How can we write the sequence of characters within ()?

- We let λ denote the **empty string**
- Then $\{a,b\}^0 = \{\lambda\}$
- $|\lambda| = 0$
- In programming, we usually denote the empty string with "" or ''

Note: The empty string can be formed over any alphabet, Σ

- Just take 0 characters from Σ to form λ

String Operations

We can define a number of interesting operations on strings:

- Concatenation
- Substring
- Prefix
- etc…

Strings: Concatenation

The **concatenation** of two strings ***s*** and ***t*** is formed by taking all symbols in ***s*** followed by all symbols in ***t***.

Concatenation of ***s*** and ***t*** is denoted by ***st***.

Formally, if $s = s_1 s_2 s_3 \ldots s_m$ and $t = t_1 t_2 t_3 \ldots t_n$ then:

$$st = s_1 s_2 s_3 \ldots s_m t_1 t_2 t_3 \ldots t_n$$

Note: $|st| = |s| + |t|$

Strings: Concatenation

Let ***s*** = cat, ***t*** = dog

st = catdog

|***st***| = 6

Let ***s*** = sponge, ***t*** = bob

st = spongebob

|***st***| = 9

sλ = λ***s*** = sponge, |λ***s***| = |***s***| = 6

Strings: Substrings

A string ***t*** is a **substring** of ***s*** if all characters of ***t*** appear consecutively in ***s***

A **prefix** of ***s*** is a substring of ***s*** that begins at the first character of ***s***

A **proper substring** of ***s*** is a substring of ***s*** that is not equal to ***s***

Strings: Substrings

Let ***s*** = racecar, ***t*** = car, ***u*** = race, ***v*** = rar, then:

1. ***s*** is a substring of ***s*** and ***s*** is a prefix of ***s***
    - ***s*** is not a proper substring of ***s***
2. ***t*** is a proper substring of ***s***
3. ***u*** is a proper substring of ***s*** and a prefix of ***s***
4. ***v*** is not a substring of ***s***

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Pairwise Disjoint Sets

Two sets A and B are **disjoint** iff $A \cap B = \varnothing$

A sequence of sets, $A_1, A_2, A_3, \ldots, A_n$ are **pairwise disjoint** if:
for any $i, j \in \{1,2,3,\ldots,n\}$, where $i \neq j$, we have $A_i \cap A_j = \varnothing$

Symbolically we write $\forall i,j \in \{1,2,3,\ldots,n\}: [(i \neq j) \rightarrow (A_i \cap A_j = \varnothing)]$

Examples

Consider the following sets:

A_1 = **{Halloween, Scream}**
A_2 = **{Frankenstein, Dracula}**
A_3 = **{Get Out, Us}**

Are these sets pairwise disjoint?

$A_1 \cap A_2 = \varnothing$ $A_2 \cap A_3 = \varnothing$ $A_1 \cap A_3 = \varnothing$

So, **yes**, all pairs are disjoint, therefore A_1, A_2, A_3 are pairwise disjoint

Examples

Consider the following sets:

$B_1 = \{ x \mid x \in \mathbb{Z}^+ \text{ and } x \text{ is even} \}$

$B_2 = \{ x \mid x \text{ is prime} \}$

$B_3 = \mathbb{Z} - \mathbb{Z}^+$

Are these sets pairwise disjoint?

Examples

Consider the following sets:

$B_1 = \{ x \mid x \in \mathbb{Z}^+ \text{ and } x \text{ is even} \}$

$B_2 = \{ x \mid x \text{ is prime} \}$

$B_3 = \mathbb{Z} - \mathbb{Z}^+$

Are these sets pairwise disjoint?

$B_1 \cap B_2 = \{2\}$ $B_2 \cap B_3 = \varnothing$ $B_1 \cap B_3 = \varnothing$

So no, the sets are not pairwise disjoint

Partitions

A **partition** of a non-empty set $\boldsymbol{A}$ is a list of one or more non-empty subsets of $\boldsymbol{A}$ such that each element of $\boldsymbol{A}$ appears in exactly one of the subsets.

Formally, a partition of $\boldsymbol{A}$ is a list of sets, $A_1, A_2, \ldots, A_k$ such that:

1. $\forall i \in [1, k]: A_i \neq \varnothing$ (the sets are non-empty)
2. $\forall i \in [1, k]: A_i \subseteq A$ (the sets are subsets of $\boldsymbol{A}$)
3. $\forall i,j \in [1, k]: i \neq j \rightarrow A_i \cap A_j = \varnothing$ (the sets are pairwise disjoint)
4. $A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k$

Partitioning Example

Consider a standard deck of cards.

- There are 52 cards.
- Each card is one of four suits: Clubs, Diamonds, Hearts, Spades
- Each suit consists of A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q
- Diamonds and Hearts are Red
- Clubs and Spades are Black

Let ***D*** be the set containing all cards in a standard deck of cards

Partitioning Example

Let $\boldsymbol{D}$ be the set containing all cards in a standard deck of cards

Let $\boldsymbol{S}_{\boldsymbol{club}}$ = { $\boldsymbol{c}$ | $\boldsymbol{c}$ is a club }
$\boldsymbol{s}_{\boldsymbol{diamond}}$ = { $\boldsymbol{c}$ | $\boldsymbol{c}$ is a diamond }
$\boldsymbol{S}_{\boldsymbol{heart}}$ = { $\boldsymbol{c}$ | $\boldsymbol{c}$ is a heart }
$\boldsymbol{S}_{\boldsymbol{spade}}$ = { $\boldsymbol{c}$ | $\boldsymbol{c}$ is a spade }

Does $\boldsymbol{S}_{\boldsymbol{club}}$, $\boldsymbol{S}_{\boldsymbol{diamond}}$, $\boldsymbol{S}_{\boldsymbol{heart}}$, $\boldsymbol{S}_{\boldsymbol{spade}}$ partition $\boldsymbol{D}$?

Partitioning Example

Let D be the set containing all cards in a standard deck of cards

Let C_{red} = { c | c is a red card }
C_{black} = { c | c is a black card }

Does C_{red}, C_{black} partition D?

Does C_{red}, S_{spade}, S_{club} partition D?

Does C_{red}, S_{heart}, S_{club} partition D?

Partitioning Example

Let ***D*** be the set containing all cards in a standard deck of cards

Let C_{red} = { **c** | **c** is a red card }
C_{black} = { **c** | **c** is a black card }

Does C_{red}, C_{black} partition ***D***? **YES**

Does C_{red}, S_{spade}, S_{club} partition ***D***? **YES**

Does C_{red}, S_{heart}, S_{club} partition ***D***? **NO. "3 of Hearts" is part of two of the sets for example. "5 of spades" is in *D* but not in any of the listed sets.**

Partitioning Example

Consider the following sets:

$\boldsymbol{O} = \{ \boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{Z} \text{ and } \boldsymbol{x} \text{ is odd} \}$

$\boldsymbol{E} = \{ \boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{Z} \text{ and } \boldsymbol{x} \text{ is even} \}$

Do $\boldsymbol{O}$ and $\boldsymbol{E}$ partition $\mathbb{N}$?

Do $\boldsymbol{O}$ and $\boldsymbol{E}$ partition $\mathbb{Z}$?

Do $\boldsymbol{O}$ and $\boldsymbol{E}$ partition $\mathbb{R}$?

Partitioning Example

Consider the following sets:

$\boldsymbol{O} = \{ \boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{Z} \text{ and } \boldsymbol{x} \text{ is odd} \}$

$\boldsymbol{E} = \{ \boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{Z} \text{ and } \boldsymbol{x} \text{ is even} \}$

Do ***O*** and ***E*** partition $\mathbb{N}$? **No**, $-1 \in (\boldsymbol{O} \cup \boldsymbol{E})$ but $-1 \notin \mathbb{N}$

Do ***O*** and ***E*** partition $\mathbb{Z}$? **Yes**.

Do ***O*** and ***E*** partition $\mathbb{R}$? **No**, $½ \notin (\boldsymbol{O} \cup \boldsymbol{E})$ but $½ \in \mathbb{N}$

Partitioning Exercise

Consider the following sets:

$\boldsymbol{A} = \{1,2,6\}$

$\boldsymbol{B} = \{2,3,4\}$

$\boldsymbol{C} = \{5\}$

$\boldsymbol{D} = \{\, \boldsymbol{x} \in \mathbb{Z}\text{: } \mathbf{1} \leq \boldsymbol{x} \leq \mathbf{6} \,\}$

Do ***A***, ***B***, ***C*** form a partition of ***D***? Why or why not?

Can you define a set ***X*** such that ***A***, ***C***, ***X*** partition ***D***?