Written Assignment #1

Content Covered: Summations, Asymptotic Bounds

Submission Process, Late Policy and Grading

Due Date: 3/1/23 @ 11:59PM
Total points: 30

Your written solution may be either handwritten and scanned, or typeset. Either way, you must produce a PDF that is legible and displays reasonably on a typical PDF reader. This PDF should be submitted via autolab as WA1. You should view your submission after you upload it to make sure that it is not corrupted or malformed. Submissions that are rotated, upside down, or that do not load will not receive credit. Illegible submissions may also lose credit depending on what can be read. Ensure that your final submission contains all pages.

You are responsible for making sure your submission went through successfully.

Written submissions may be turned in up to one day late for a 50% penalty.
No grace day usage is allowed.

Problem 1 - Summations

[21/30 points]

Consider the following 6 functions.

$$f_1(n) = \sum_{i=n}^{2n} 6i$$

$$f_2(n) = \sum_{i=1}^{5} i^3$$

$$f_3(n) = \sum_{i=0}^{n} 4\log(2^i \cdot 2^{2^i})$$

$$f_4(n) = \sum_{i=1}^{n} \sum_{j=2}^{log(i)-2} 2^j$$

$$f_5(n) = \sum_{i=n}^{n^2} 3i$$

$$f_6(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} i$$

Part A

[2 points per function, total = 12 points]

For each of the six functions above, compute the closed form for the summation (a formula that has no summations or variables other than n).

To compute the closed form of each summation, you must show all work as a step-by-step derivation using the rules in the reference material below. You may only use the rules listed there (i.e, **R1** to **R10**). See the example below for specifics on how to perform the derivation.

Part B

[1 point per function, total = 6 points]

For each of the six functions above, provide a tight (i.e., Big-Θ) bound for the formula. No justification is required.

Part C

[3 points]

Arrange the functions in order of growth, from slowest growth on the left to fastest growth on the right. In the event of a tie, put the function with the smaller constant terms to the left. No justification is required.

Problem 2 - Asymptotic Bounds

[9/30 points]

Suppose $f(n) = 9n^2 - 16n + 42(\log(n))^2$ and $g(n) = 4n^2$. Using the formal definition of Big-O, prove that $f(n) = O(g(n))$ by providing valid constants c, n_0 and proving that they are valid (that the inequality holds). Verify your result by using the limit test described in lecture.

Reference Material

Closed form summation equivalences

The following works for any functions f, g (even constants). c is any constant relative to $i, j, k, \ell \in \mathbb{Z}$. Any sum $sum_{i=j}^{k} f(i)$ is always 0 if $k < j$.

R1. $\sum_{i=j}^{k} c = (k - j + 1)c$

R2. $\sum_{i=j}^{k} (cf(i)) = c \sum_{i=j}^{k} f(i)$

R3. $\sum_{i=j}^{k} (f(i) + g(i)) = \left(\sum_{i=j}^{k} f(i)\right) + \left(\sum_{i=j}^{k} g(i)\right)$

R4. $\sum_{i=j}^{k} (f(i)) = \left(\sum_{i=\ell}^{k} (f(i))\right) - \left(\sum_{i=\ell}^{j-1} (f(i))\right)$‘ (for any ‘$\ell < j$)

R5. $\sum_{i=j}^{k} f(i) = f(j) + f(j+1) + \ldots + f(k-1) + f(k)$

R6. $\sum_{i=j}^{k} f(i) = f(j) + \ldots + f(\ell - 1) + \left(\sum_{i=\ell}^{k} f(i)\right)$ (for any $j < \ell \leq k$)

R7. $\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{\ell} f(i)\right) + f(\ell + 1) + \ldots + f(k)$ (for any $j \leq \ell < k$)

R8. $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

R9. $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

R10. $n! \leq c_s n^n$ is a tight upper bound (Sterling: Some constant c_s exists)

Closed form logarithm equivalences

R11. $\log(n^a) = a \log(n)$

R12. $\log(an) = \log(a) + \log(n)$

R13. $\log(\frac{n}{a}) = \log(n) - \log(a)$

R14. $\log_b(n) = \frac{\log_c(n)}{\log_c(b)}$

R15. $\log(2^n) = 2^{\log(n)} = n$

Example

The derivation to find the closed form for $\sum_{i=0}^{n-2} \sum_{j=0}^{i} 20$ is as follows:

apply **R1** with $j = 0, k = i, c = 20$

$$\sum_{i=0}^{n-2} \sum_{j=0}^{i} 20 = \sum_{i=0}^{n-2} (i - 0 + 1)20 = \sum_{i=0}^{n-2} (i + 1)20$$

apply **R6** with $j = 0, \ell = 1, k = n - 2$

$$= 1 \cdot 20 + \sum_{i=1}^{n-2} (i + 1)20 = 20 + \sum_{i=1}^{n-2} (i + 1)20$$

apply **R2** with $c = 20, f(i) = (i + 1), j = 1, k = n - 2$

$$= 20 + 20 \sum_{i=1}^{n-2} (i + 1)$$

apply **R3** with $f(i) = i, g(i) = 1, j = 1, k = n - 2$

$$= 20 + 20 \left(\sum_{i=1}^{n-2} i + \sum_{i=1}^{n-2} 1 \right)$$

apply **R8** with $k = n - 2$

$$= 20 + 20 \left(\frac{(n-2)(n-1)}{2} + \sum_{i=1}^{n-2} 1 \right)$$

apply **R1** with $c = 1, j = 1, k = n - 2$

$$= 20 + 20 \left(\frac{(n-2)(n-1)}{2} + (n - 2 - 1 + 1) \cdot 1 \right)$$

$$= 20 + 20 \left(\frac{n^2 - 3n + 2}{2} + (n - 2) \right)$$

$$= 20 + (10n^2 - 30n + 20) + (20n - 40)$$

$$= 10n^2 - 10n$$