Question 1, Parts a and b

**Function $f_1$**

\[ \sum_{i=n}^{2n} 6i \]

Apply Rule 2 with $c = 6$

\[ = 6 \sum_{i=n}^{2n} i \]

Apply Rule 4 with $k = 2n, j = n, \ell = 1$; Assume that $n > 1$

\[ = 6 \left( \sum_{i=1}^{2n} i - \sum_{i=1}^{n-1} i \right) \]

Apply Rule 8 twice, with $k = 2n$ and $k = n - 1$ respectively

\[ = 6 \left( \frac{2n(2n + 1)}{2} - \frac{(n - 1)(n - 1 + 1)}{2} \right) \]

\[ = (12n^2 + 6n) - (3n^2 - 3n) \]

\[ = 9n^2 + 9n \]

\[ = \Theta(n^2) \]

**Function $f_2$**

\[ \sum_{i=1}^{5} i^3 \]

Apply Rule 5 with $j = 1, k = 5$

\[ = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 \]
\[
1 + 8 + 27 + 64 + 125 \\
= 225 \\
= \Theta(1)
\]

**Function** \( f_3 \)

\[
\sum_{i=0}^{n} 4 \log(2^i \cdot 2^{2^i})
\]

Apply Rule 2 with \( c = 4 \) and \( f(i) = \log(2^i \cdot 2^{2^i}) \)

\[
= 4 \sum_{i=0}^{n} \log(2^i \cdot 2^{2^i})
\]

Apply Rule 12 with \( a = 2^i \) and \( n = 2^{2^i} \)

\[
= 4 \sum_{i=0}^{n} \left( \log(2^i) + \log(2^{2^i}) \right)
\]

Apply Rule 15 twice

\[
= 4 \sum_{i=0}^{n} \left( i + 2^i \right)
\]

Apply Rule 3 with \( j = 0, k = n, f(i) = i, g(i) = 2^i \)

\[
= 4 \left( \sum_{i=0}^{n} i + \sum_{i=0}^{n} 2^i \right)
\]

Apply Rule 9 with \( k = n \)

\[
= 4 \left( \sum_{i=0}^{n} i + 2^{n+1} - 1 \right)
\]

Apply Rule 6 with \( k = n, j = 0, \ell = 1 \)

\[
= 4 \left( 0 + \sum_{i=1}^{n} i + 2^{n+1} - 1 \right)
\]
Apply Rule 8 with \( k = n \)

\[
= 4 \left( 0 + \frac{n(n + 1)}{2} + 2^{n+1} - 1 \right)
\]

\[
= 4 \left( \frac{n^2}{2} + \frac{n}{2} + 2 \cdot 2^n - 1 \right)
\]

\[
= 2n^2 + 2n + 8 \cdot 2^n - 4
\]

\[
= \Theta(2^n)
\]

**Function** \( f_4 \)

\[
\sum_{i=1}^{n} \sum_{j=2}^{\log(i)-2} 2^j
\]

Apply Rule 4 with \( k = \log(i) - 2, j = 2, \ell' = 0 \)

\[
= \sum_{i=1}^{n} \left( \sum_{j=0}^{\log(i)-2} 2^j - \sum_{j=0}^{1} 2^j \right)
\]

Apply Rule 9 with \( k = \log(i) - 2 \)

\[
= \sum_{i=1}^{n} \left( 2^{\log(i)-2+1} - 1 - \sum_{j=0}^{1} 2^j \right)
\]

\[
= \sum_{i=1}^{n} \left( \frac{i}{2} - 1 - \sum_{j=0}^{1} 2^j \right)
\]

Apply Rule 6 with \( j = 0, k = 1 \)

\[
= \sum_{i=1}^{n} \left( \frac{i}{2} - 1 - (2^0 + 2^1) \right)
\]

\[
= \sum_{i=1}^{n} \left( \frac{i}{2} - 1 - 1 - 2 \right)
\]
Apply Rule 3 with \(f(i) = \frac{i}{2}, g(i) = -4\)

\[
= \sum_{i=1}^{n} \left( \frac{i}{2} - 4 \right)
\]

Apply Rule 1 with \(j = 1, k = n, c = -4\)

\[
= \sum_{i=1}^{n} \frac{i}{2} + \sum_{i=1}^{n} (-4)
\]

Apply Rule 1 with \(j = 1, k = n, c = -4\)

\[
= \sum_{i=1}^{n} \frac{i}{2} + (n - 1 + 1)(-4)
\]

\[
= \sum_{i=1}^{n} \frac{i}{2} - 4n
\]

Apply Rule 2 with \(c = \frac{1}{2}\)

\[
= \frac{1}{2} \sum_{i=1}^{n} i - 4n
\]

Apply Rule 8 with \(k = n\)

\[
= \frac{1}{2} \left( \frac{n(n+1)}{2} \right) - 4n
\]

\[
= \frac{n^2}{4} + \frac{n}{4} - 4n
\]

\[
= \frac{n^2}{4} - \frac{15n}{4}
\]

\[
= \Theta(n^2)
\]

**Function \(f_5\)**

\[
\sum_{i=n}^{n^2} 3i
\]

Apply Rule 2 with \(c = 3, f(i) = i\)
Apply Rule 4 with \( j = n, k = n^2, \ell = 1 \); Assume \( n > 1 \)

\[
= 3 \sum_{i=n}^{n^2} i
\]

Apply Rule 8 with \( k = n^2 \)

\[
= 3 \left( \sum_{i=1}^{n^2} i - \sum_{i=1}^{n-1} i \right)
\]

Apply Rule 8 with \( k = n - 1 \)

\[
= 3 \left( \frac{n^2(n^2 + 1)}{2} - \frac{(n - 1)(n - 1 + 1)}{2} \right)
\]

\[
= 3 \left( \frac{n^4}{2} + \frac{n^2}{2} - \frac{n^2}{2} + \frac{n}{2} \right)
\]

\[
= \frac{3n^4}{2} + \frac{3n}{2}
\]

\[
= \Theta(n^4)
\]

**Function** \( f_6 \)

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} i
\]

Apply Rule 1 with \( j = 1, k = n, c = i \)

\[
= \sum_{i=1}^{n} (n - 1 + 1)i
\]

\[
= \sum_{i=1}^{n} (n - 1 + 1)i
\]

Apply Rule 2 with \( c = n, f(i) = i \)
Apply Rule 8 with $k = n$

$$= n \sum_{i=1}^{n} i$$

$$= n \cdot \frac{n(n + 1)}{2}$$

$$= \frac{n^3}{2} + \frac{n^2}{2}$$

$$= \Theta(n^3)$$

**Question 1, Part c**

1. $f_2 = \Theta(1)$
2. $f_4 = \Theta(n^2)$ (with a constant of $\frac{1}{4}$)
3. $f_1 = \Theta(n^2)$ (with a constant of 9)
4. $f_6 = \Theta(n^3)$
5. $f_5 = \Theta(n^4)$
6. $f_3 = \Theta(2^n)$

**Question 2**

$$f(n) = 9n^2 - 16n + 42(\log(n))^2$$

$$g(n) = 4n^2$$

To Show: There is a $c > 0$, $n_0 > 0$ such that for any $n \geq n_0$:

$$9n^2 - 16n + 42(\log(n))^2 \leq c4n^2$$

Define new variables $x, y, z$ such that $x + y + z = c$. The above objective is equivalent to showing that we can pick $x, y, z$ that satisfy $x + y + z > 0$ such that the following 3 inequality hold:

1. $9n^2 \leq x4n^2$
2. $-16n \leq y4n^2$
3. $42 \log^2(n) \leq z4n^2$

**Inequality 1**
If we set \( x = 3 \) then \( 9n^2 \leq 12n^2 \), and this is given

**Inequality 2**

\[-16n \leq y4n^2\]

This is true for any \( n > 0, y \geq 0 \). Arbitrarily pick \( y = 1 \)

**Inequality 3**

Subgoal: If we can show that \( 42 \log^2(n) \leq n \log(n) \leq z4n^2 \), then by transitivity, we have inequality \#3.

**Step 1:**

\[42 \log^2(n) \leq n \log(n)\]

Divide both sides by \( \log(n) \); Pick \( n_0 > 1 \) to ensure that \( \log(n) > 0 \)

\[42 \log(n) \leq n\]

This equivalence is given for sufficiently large \( n \)

**Step 2:**

\[n \log(n) \leq z4n^2\]

Divide both sides by \( n \); Pick \( n_0 > 0 \) to make this valid

\[\log(n) \leq z4n\]

This equivalence is given for sufficiently large \( n \) if we set \( z = 1 \)

Thus, for sufficiently large \( n \), we have \( 42 \log^2(n) \leq z4n^2 \) for \( z = 1 \)

**Conclusion**

\[c = x + y + z = 3 + 1 + 1 = 5\], so we have shown that, for sufficiently large \( n \):

\[9n^2 - 16n + 42(\log(n))^2 \leq 5(4n^2)\]

**Limit Test**
Therefore since $\lim_{n \to \infty} \frac{f(n)}{g(n)}$ is a constant, then $f(n) \in \Theta(g(n))$