Question 1, Parts a and b

Function f_1

$$\sum_{i=n}^{2n} 6i$$

Apply Rule 2 with $c = 6$

$$= 6 \sum_{i=n}^{2n} i$$

Apply Rule 4 with $k = 2n, j = n, \ell = 1$; Assume that $n > 1$

$$= 6 \left(\sum_{i=1}^{2n} i - \sum_{i=1}^{n-1} i \right)$$

Apply Rule 8 twice, with $k = 2n$ and $k = n - 1$ respectively

$$= 6 \left(\frac{2n(2n+1)}{2} - \frac{(n-1)(n-1+1)}{2} \right)$$

$$= (12n^2 + 6n) - (3n^2 - 3n)$$

$$= 9n^2 + 9n$$

$$= \Theta(n^2)$$

Function f_2

$$\sum_{i=1}^{5} i^3$$

Apply Rule 5 with $j = 1$, $k = 5$

$$= 1^3 + 2^3 + 3^3 + 4^3 + 5^3$$

$$= 1 + 8 + 27 + 64 + 125$$

$$= 225$$

$$= \Theta(1)$$

Function f_3

$$\sum_{i=0}^{n} 4 \log(2^i \cdot 2^{2^i})$$

Apply Rule 2 with $c = 4$ and $f(i) = \log(2^i \cdot 2^{2^i})$

$$= 4 \sum_{i=0}^{n} \log(2^i \cdot 2^{2^i})$$

Apply Rule 12 with $a = 2^i$ and $n = 2^{2^i}$

$$= 4 \sum_{i=0}^{n} \left(\log(2^i) + \log(2^{2^i})\right)$$

Apply Rule 15 twice

$$= 4 \sum_{i=0}^{n} \left(i + 2^i\right)$$

Apply Rule 3 with $j = 0, k = n, f(i) = i, g(i) = 2^i$

$$= 4 \left(\sum_{i=0}^{n} i + \sum_{i=0}^{n} 2^i \right)$$

Apply Rule 9 with $k = n$

$$= 4 \left(\sum_{i=0}^{n} i + 2^{n+1} - 1 \right)$$

Apply Rule 6 with $k = n, j = 0, \ell = 1$

$$= 4 \left(0 + \sum_{i=1}^{n} i + 2^{n+1} - 1 \right)$$

Apply Rule 8 with $k = n$

$$= 4\left(0 + \frac{n(n+1)}{2} + 2^{n+1} - 1\right)$$

$$= 4\left(\frac{n^2}{2} + \frac{n}{2} + 2 \cdot 2^n - 1\right)$$

$$= 2n^2 + 2n + 8 \cdot 2^n - 4$$

$$= \Theta(2^n)$$

Function f_4

$$\sum_{i=1}^{n} \sum_{j=2}^{\log(i)-2} 2^j$$

Apply Rule 4 with $k = \log(i) - 2, j = 2, \ell = 0$

$$= \sum_{i=1}^{n} \left(\sum_{j=0}^{\log(i)-2} 2^j - \sum_{j=0}^{1} 2^j \right)$$

Apply Rule 9 with $k = \log(i) - 2$

$$= \sum_{i=1}^{n} \left(2^{\log(i)-2+1} - 1 - \sum_{j=0}^{1} 2^j \right)$$

$$= \sum_{i=1}^{n} \left(2^{\log(i)-1} - 1 - \sum_{j=0}^{1} 2^j \right)$$

$$= \sum_{i=1}^{n} \left(\frac{i}{2} - 1 - \sum_{j=0}^{1} 2^j \right)$$

Apply Rule 6 with $j = 0, k = 1$

$$= \sum_{i=1}^{n} \left(\frac{i}{2} - 1 - (2^0 + 2^1) \right)$$

$$= \sum_{i=1}^{n} \left(\frac{i}{2} - 1 - 1 - 2 \right)$$

$$= \sum_{i=1}^{n} \left(\frac{i}{2} - 4 \right)$$

Apply Rule 3 with $f(i) = \frac{i}{2}$, $g(i) = -4$

$$= \sum_{i=1}^{n} \frac{i}{2} + \sum_{i=1}^{n} (-4)$$

Apply Rule 1 with $j = 1, k = n, c = -4$

$$= \sum_{i=1}^{n} \frac{i}{2} + (n - 1 + 1)(-4)$$

$$= \sum_{i=1}^{n} \frac{i}{2} - 4n$$

Apply Rule 2 with $c = \frac{1}{2}$

$$= \frac{1}{2} \sum_{i=1}^{n} i - 4n$$

Apply Rule 8 with $k = n$

$$= \frac{1}{2} \left(\frac{n(n+1)}{2} \right) - 4n$$

$$= \frac{n^2}{4} + \frac{n}{4} - 4n$$

$$= \frac{n^2}{4} - \frac{15n}{4}$$

$$= \Theta(n^2)$$

Function f_5

$$\sum_{i=n}^{n^2} 3i$$

Apply Rule 2 with $c = 3, f(i) = i$

$$= 3 \sum_{i=n}^{n^2} i$$

Apply Rule 4 with $j = n$, $k = n^2$, $\ell = 1$; Assume $n > 1$

$$= 3 \left(\sum_{i=1}^{n^2} i - \sum_{i=1}^{n-1} i \right)$$

Apply Rule 8 with $k = n^2$

$$= 3 \left(\frac{n^2 (n^2 + 1)}{2} - \sum_{i=1}^{n-1} i \right)$$

Apply Rule 8 with $k = n - 1$

$$= 3 \left(\frac{n^2 (n^2 + 1)}{2} - \frac{(n-1)(n-1+1)}{2} \right)$$

$$= 3 \left(\frac{n^4}{2} + \frac{n^2}{2} - \frac{n^2}{2} + \frac{n}{2} \right)$$

$$= \frac{3n^4}{2} + \frac{3n}{2}$$

$$= \Theta(n^4)$$

Function f_6

$$\sum_{i=1}^{n} \sum_{j=1}^{n} i$$

Apply Rule 1 with $j = 1$, $k = n$, $c = i$

$$= \sum_{i=1}^{n} (n - 1 + 1) i$$

$$= \sum_{i=1}^{n} ni$$

Apply Rule 2 with $c = n, f(i) = i$

$$= n \sum_{i=1}^{n} i$$

Apply Rule 8 with $k = n$

$$= n\frac{n(n+1)}{2}$$

$$= \frac{n^3}{2} + \frac{n^2}{2}$$

$$= \Theta(n^3)$$

Question 1, Part c

1. $f_2 = \Theta(1)$
2. $f_4 = \Theta(n^2)$ (with a constant of $\frac{1}{4}$)
3. $f_1 = \Theta(n^2)$ (with a constant of 9)
4. $f_6 = \Theta(n^3)$
5. $f_5 = \Theta(n^4)$
6. $f_3 = \Theta(2^n)$

Question 2

$$f(n) = 9n^2 - 16n + 42(\log(n))^2$$

$$g(n) = 4n^2$$

To Show: There is a $c > 0$, $n_0 > 0$ such that for any $n \geq n_0$:

$$9n^2 - 16n + 42(\log(n))^2 \leq c4n^2$$

Define new variables x, y, z such that $x + y + z = c$. The above objective is equivalent to showing that we can pick x, y, z that satisfy $x + y + z > 0$ such that the following 3 inequality hold:

1. $9n^2 \leq x4n^2$
2. $-16n \leq y4n^2$
3. $42\log^2(n) \leq z4n^2$

Inequality 1

$$9n^2 \leq x4n^2$$

If we set $x = 3$ then $9n^2 \leq 12n^2$, and this is given

Inequality 2

$$-16n \leq y4n^2$$

This is true for any $n > 0$, $y \geq 0$. Arbitrarily pick $y = 1$

Inequality 3

Subgoal: If we can show that $42\log^2(n) \leq n\log(n) \leq z4n^2$, then by transitivity, we have inequality #3.

Step 1:

$$42\log^2(n) \leq n\log(n)$$

Divide both sides by $\log(n)$; Pick $n_0 > 1$ to ensure that $\log(n) > 0$

$$42\log(n) \leq n$$

This equivalence is given for sufficiently large n

Step 2:

$$n\log(n) \leq z4n^2$$

Divide both sides by n; Pick $n_0 > 0$ to make this valid

$$\log(n) \leq z4n$$

This equivalence is given for sufficiently large n if we set $z = 1$

Thus, for sufficiently large n, we have $42\log^2(n) \leq z4n^2$ for $z = 1$

Conclusion

$c = x + y + z = 3 + 1 + 1 = 5$, so we have shown that, for sufficiently large n:

$$9n^2 - 16n + 42(\log(n))^2 \leq 5(4n^2)$$

Limit Test

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{9n^2 - 16n + 42(\log(n))^2}{4n^2}$$

$$= \lim_{n\to\infty} \left(\frac{9}{4} - \frac{4}{n} + \frac{42(log(n))^2}{4n^2} \right) = \frac{9}{4} - 0 + 0$$

Therefore since $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is a constant, then $f(n) \in \Theta(g(n))$