CSE 250

Data Structures

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Runtime Analysis - Examples

Announcements

- PA1 is out, due on Sunday
  - Autolab will be up by tonight

Recap of Runtime Complexity

Big-Θ

- Growth functions are in the **same** complexity class
- If f(n) ∈ **Θ**(g(n)) then an algorithm taking f(n) steps is as "exactly" as fast as one that takes g(n) steps.

Big-O

- Growth functions in the **same or smaller** complexity class.
- If f(n) ∈ O(g(n)), then an algorithm that takes f(n) steps is *at least as fast* as one taking g(n) (but it may be even faster).

Big-Ω

- Growth functions in the **same or bigger** complexity class
- If f(n) ∈ **Ω**(g(n)), then an algorithm that takes f(n) steps is *at least as slow* as one that takes g(n) steps (but it may be even slower)

Recap of Runtime Complexity

Big-Θ — Tight Bound

- Growth functions are in the **same** complexity class
- If f(n) ∈ **Θ**(g(n)) then an algorithm taking f(n) steps is as "exactly" as fast as one that takes g(n) steps.

Big-O — Upper Bound

- Growth functions in the **same or smaller** complexity class.
- If f(n) ∈ O(g(n)), then an algorithm that takes f(n) steps is *at least as fast* as one taking g(n) (but it may be even faster).

Big-Ω — Lower Bound

- Growth functions in the **same or bigger** complexity class
- If f(n) ∈ **Ω**(g(n)), then an algorithm that takes f(n) steps is *at least as slow* as one that takes g(n) steps (but it may be even slower)

Common Runtimes (in order of complexity)

Constant Time: $\Theta(1)$

Logarithmic Time: $\Theta(\log(n))$

Linear Time: $\Theta(n)$

Quadratic Time: $\Theta(n^2)$

Polynomial Time: $\Theta(n^k)$ for some $k > 0$

Exponential Time: $\Theta(c^n)$ (for some $c \geq 1$)

Common Runtimes (in order of complexity)

Constant Time:	$\Theta(1)$	$T(n) = c$
Logarithmic Time:	$\Theta(\log(n))$	$T(n) = c \log(n)$
Linear Time:	$\Theta(n)$	$T(n) = c_1 n + c_0$
Quadratic Time:	$\Theta(n^2)$	$T(n) = c_2 n^2 + c_1 n^1 + c_0$
Polynomial Time:	$\Theta(n^k)$ for some $k > 0$	$T(n) = c_k n^k + \ldots + c_1 n + c_0$
Exponential Time:	$\Theta(c^n)$ (for some $c \geq 1$)	$T(n) = c^n$

Constants vs Asymptotics

Given the following pseudocode:

```
for (i ← 0 until n) { /* do work */ }
```

If the `/* do work */` portion of the code originally takes 10 steps…

But we optimize it to now take 7 steps…

Our total runtime goes from 10n steps to 7n steps: 30% faster!

…but still **Θ(n)**

c and n_0

Compare the two runtimes:

$$T_1(n) = 100n$$

$$T_2(n) = n^2$$

- $100n \in O(n^2)$ (T_2 is the slower runtime)
- …but consider if $c_{high} = 1, n_0 = 100$
- Until our input size reaches 100 or more, T_2 is the **faster** runtime

Takeaways

Asymptotically slower runtimes *can* be better in real-world situations.

- An algorithm with runtime T_2 is better on small inputs
- An algorithm with runtime T_2 might be easier to implement/maintain
- An algorithm with runtime T_1 might not exists

Takeaways

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(sometimes this is provable…see CSE 331)

Takeaways

The important thing is learning the tools to *reason* about the different algorithms and *why* you might choose one over the other!

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The important thing is learning the tools to *reason* about the different algorithms and *why* you might choose one over the other!

…But for this class, we can assume that if $T_2(n)$ is in a bigger complexity class, then $T_1(n)$ is better/faster/stronger.

Now some examples…
…and common pitfalls

Bubble Sort

```
bubblesort(seq: Seq[Int]):
  n ← seq length
  for i ← n-2 to 0, by -1:
    for j ← i to n-1:
      if seq(j+1) < seq(j):
        swap seq(j) and seq(j+1)
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What is the runtime complexity class for bubblesort?

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How many steps does it take?

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How times does it execute lines 5 and 6?

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This loop has 1 iteration when i = n-2
…2 iterations when i = n-3
…3 iterations when i = n-4
…
n-1 iterations when i = 0

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How times does it execute lines 5 and 6? 1 + 2 + 3 + 4 + 5 … + n-1

Helpful Summation Rules

1. $\sum_{i=j}^{k} c = (k - j + 1)c$

2. $\sum_{i=j}^{k}(cf(i)) = c \sum_{i=j}^{k} f(i)$

3. $\sum_{i=j}^{k}(f(i) + g(i)) = \left(\sum_{i=j}^{k} f(i)\right) + \left(\sum_{i=j}^{k} g(i)\right)$

4. $\sum_{i=j}^{k}(f(i)) = \left(\sum_{i=\ell}^{k}(f(i))\right) - \left(\sum_{i=\ell}^{j-1}(f(i))\right)$ (for any $\ell < j$)

5. $\sum_{i=j}^{k} f(i) = f(j) + f(j+1) + \ldots + f(k-1) + f(k)$

6. $\sum_{i=j}^{k} f(i) = f(j) + \ldots + f(\ell - 1) + \left(\sum_{i=\ell}^{k} f(i)\right)$ (for any $j < \ell \leq k$)

7. $\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{\ell} f(i)\right) + f(\ell + 1) + \ldots + f(k)$ (for any $j \leq \ell < k$)

8. $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

9. $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

10. $n! \leq c_s n^n$ is a tight upper bound (Sterling: Some constant c_s exists)

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$$1 + 2 + 3 + 4 + 5 \ldots + n-1 = (n) * (n - 1) / 2 = \boldsymbol{O(n^2)}$$

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What is the complexity of this step?
Do not assume function calls take O(1) time!

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Searching Sequences

```
def indexOf[T](seq: Seq[T], value: T, from: Int): Int = {
    for(i <- from until seq.length) {
      if(seq(i).equals(value)) { return i }
    }
    return -1
}
```

What is the complexity?

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What is the complexity? O(n)

Searching Sequences

```
def count[T](seq: Seq[T], value: T): Int ={
    var count = 0;
    var i = indexOf(seq, value, 0)
    while(i != -1) {
        count += 1;
        i = indexOf(seq, value, i+1)
    }
    return count
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What is the complexity? Each element is only checked once, so O(n).

Searching Sorted Sequences

- Assuming O(1) access to elements ('random access')
  - Divide the set of elements in half by taking the "middle" element, *m*
    - If *m* is greater than what we are looking for, search the lower half
    - If *m* is less than what we are looking for, search the right half
    - Repeat until you've found the element or you can't divide in half

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Therefore the runtime of this search algorithm is $O(\log(n))$