CSE 250
Data Structures

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Recursion
Textbook Ch 15

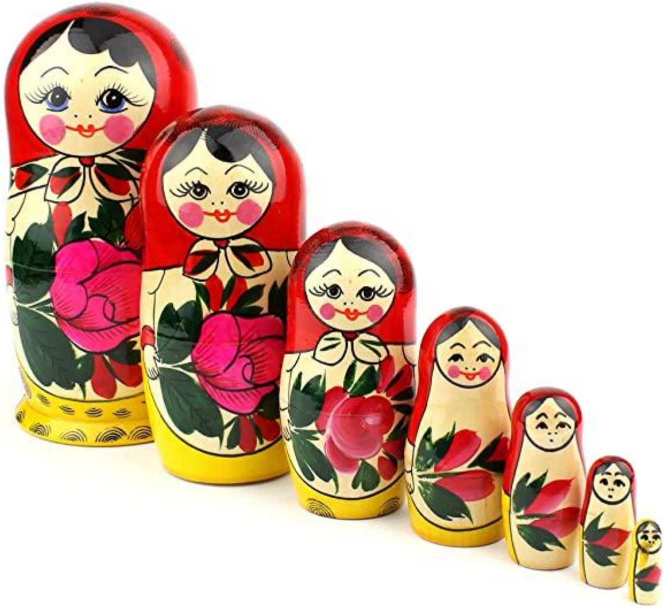
Announcements

- WA1 posted. Due Wed 3/1/23.

`Seq` Summary So Far

Operation	`Array[T]`	`ArrayBuffer[T]`	`List[T]` *(index)*	`List[T]` *(ref)*
`apply(i)`	$\Theta(1)$	$\Theta(1)$	$\Theta(i)$, $O(n)$	$\Theta(1)$
`update(i, val)`	$\Theta(1)$	$\Theta(1)$	$\Theta(i)$, $O(n)$	$\Theta(1)$
`insert(i, val)`	$\Theta(n)$	$O(n)$	$\Theta(i)$, $O(n)$	$\Theta(1)$
`remove(i, val)`	$\Theta(n)$	$\Theta(n-i)$, $O(n)$	$\Theta(i)$, $O(n)$	$\Theta(1)$
`append(i)`	$\Theta(n)$	$O(n)$, Amortized $\Theta(1)$	$\Theta(i)$, $O(n)$	$\Theta(1)$

Recursion



Factorial

factorial(n) = n * (n-1) * (n-2) * … * 2 * 1

Factorial

$$\text{factorial}(n) = n * \underbrace{(n-1) * (n-2) * \ldots * 2 * 1}_{\text{factorial}(n-1)}$$

Factorial

$$\text{factorial}(n) = n * \underbrace{(n-1) * \underbrace{(n-2) * \ldots * 2 * 1}_{\text{factorial}(n-2)}}_{\text{factorial}(n-1)}$$

Factorial

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Factorial

$$\text{factorial}(n) = n * \underbrace{(n-1) * \underbrace{(n-2) * \dots * \underbrace{2 * \overbrace{1}^{\text{factorial}(1)}}_{\text{factorial}(2)}}_{\text{factorial}(n-2)}}_{\text{factorial}(n-1)}$$

Factorial

```
def factorial(n: Int): Long =
    if(n <= 1){ 1 }
    else { n * factorial(n - 1) }
```

Factorial

```
def factorial(n: Int): Long =
    if(n <= 1){ 1 }                    ← Base Case
    else { n * factorial(n - 1) }
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Factorial

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def factorial(n: Int): Long =
    if(n <= 1){ 1 }                         ← Base Case
    else { n * factorial(n - 1) } ← Recursive Case
```

Fibonacci

fibb(n) = 1, 1

Fibonacci

$$\text{fibb}(n) = \underbrace{1, 1}_{+}, \boxed{2}$$

Fibonacci

$$\text{fibb}(n) = 1,\ \underbrace{1,\ 2}_{+},\ \boxed{3}$$

Fibonacci

$$\text{fibb}(n) = 1, 1, \underbrace{2, 3}_{+}, \boxed{5}$$

Fibonacci

fibb(n) = 1, 1, 2, 3, 5, 8, 13, 21, 34, …

Fibonacci

fibb(n) = 1, 1, 2, 3, 5, 8, 13, 21, 34, …

fibb(n) = fibb(n-1) + fibb(n-2)

Fibonacci

```
def fibb(n: Int): Long =
    if(n < 2){ 1 }
    else { fibb(n-1) + fibb(n-2) }
```

Fibonacci

```
def fibb(n: Int): Long =
    if(n < 2){ 1 }                          ← Base Case
    else { fibb(n-1) + fibb(n-2) }
```

Fibonacci

```
def fibb(n: Int): Long =
    if(n < 2){ 1 }                            ← Base Case
    else { fibb(n-1) + fibb(n-2) }            ← Recursive Case
```

Towers of Hanoi

Live demo!

But What is the Complexity?

```
def factorial(n: Int): Long =
    if(n <= 1){ 1 }
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def factorial(n: Int): Long =
    if(n <= 1){ 1 }                          ← Θ(1)
    else { n * factorial(n - 1) }
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def factorial(n: Int): Long =
    if(n <= 1){ 1 }                          ← Θ(1)
    else { n * factorial(n - 1) }           ← Θ(1) + ???
```

But What is the Complexity?

```
def factorial(n: Int): Long =
    if(n <= 1){ 1 }                        ← Θ(1)
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How do we figure out complexity of a function, when part of the runtime of the function is calling itself?

But What is the Complexity?

```
def factorial(n: Int): Long =
    if(n <= 1){ 1 }                    ← Θ(1)
    else { n * factorial(n - 1) } ← Θ(1) + ???
```

How do we figure out complexity of a function, when part of the runtime of the function is calling itself?

To know the complexity of **`factorial`**, *we need to…know the complexity of* **`factorial`**?

Complexity of `factorial`

$$T(n) = \begin{cases} \Theta(1) & \textbf{if } n \leq 1 \\ T(n-1) + \Theta(1) & \textbf{otherwise} \end{cases}$$

Solve for $T(n)$

Complexity of `factorial`

Solve for $T(n)$

Approach:

1. Generate a hypothesis
2. Prove your hypothesis for the base case
3. Prove the hypothesis for the recursive case *inductively*

Step 1 - Generate a Hypothesis

Let's start by looking at the runtime for increasing values of *n*

$\Theta(1)$

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What is the pattern?

Step 1 - Generate a Hypothesis

Let's start by looking at the runtime for increasing values of n

$\Theta(1), 2\Theta(1), 3\Theta(1), 4\Theta(1), 5\Theta(1), 6\Theta(1), 7\Theta(1)$

What is the pattern?

Hypothesis: $T(n) \in O(n)$

(there is some $c > 0$ such that $T(n) \leq c \cdot n$)

Prove for the Base Case

First, lets make our constants explicit

$$T(n) = \begin{cases} c_0 & \textbf{if } n \leq 1 \\ T(n-1) + c_1 & \textbf{otherwise} \end{cases}$$

Prove $T(n) \in O(n)$ for the Base Case

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case: n = 1

$$T(1) \leq c \cdot 1$$

Prove $T(n) \in O(n)$ for the Base Case

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Base Case: n = 1

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$$T(1) \leq c$$

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Base Case: n = 1

$$T(1) \leq c \cdot 1$$

$$T(1) \leq c$$

$$c_0 \leq c$$

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Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case: n = 1

$$T(1) \leq c \cdot 1$$

$$T(1) \leq c$$

$$c_0 \leq c$$

True for any $c \geq c_0$

Prove $T(n) \in O(n)$ for the Base Case + 1

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 1: n = 2

$$T(2) \leq c \cdot 2$$

Prove $T(n) \in O(n)$ for the Base Case + 1

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \le c \cdot n$)
Base Case + 1: n = 2

$$T(2) \le c \cdot 2$$

$$T(1) + c_1 \le 2c$$

Prove $T(n) \in O(n)$ for the Base Case + 1

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Base Case + 1: n = 2

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$$T(1) + c_1 \leq 2c$$

$$c_0 + c_1 \leq 2c$$

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$$T(1) + c_1 \leq 2c$$

$$c_0 + c_1 \leq 2c$$

We already know there's a $c \geq c_0$, so…

Prove $T(n) \in O(n)$ for the Base Case + 1

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True for any $c \geq c_1$

Prove $T(n) \in O(n)$ for the Base Case + 2

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)
Base Case + 2: n = 3

$$T(3) \leq c \cdot 3$$

Prove $T(n) \in O(n)$ for the Base Case + 2

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)

Base Case + 2: n = 3

$$T(3) \leq c \cdot 3$$

$$T(2) + c_1 \leq 3c$$

Prove $T(n) \in O(n)$ for the Base Case + 2

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Base Case + 2: n = 3

$$T(3) \leq c \cdot 3$$

$$T(2) + c_1 \leq 3c$$

We know there's a c s.t. $T(2) \leq 2c$,

So if we show that $2c + c_1 \leq 3c$, then $T(2) + c_1 \leq 2c + c_1 \leq 3c$

Prove $T(n) \in O(n)$ for the Base Case + 2

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)
Base Case + 2: n = 3

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We know there's a c s.t. $T(2) \leq 2c$,

So if we show that $2c + c_1 \leq 3c$, then $T(2) + c_1 \leq 2c + c_1 \leq 3c$

True for any $c \geq c_1$

Prove $T(n) \in O(n)$ for the Base Case + 3

Prove: $T(n) \in O(n)$ (ie: there exists a constant, c, such that $T(n) \leq c \cdot n$)
Base Case + 2: n = 4

$$T(4) \leq c \cdot 4$$

$$T(3) + c_1 \leq 4c$$

We know there's a c s.t. $T(3) \leq 3c$,

So if we show that $3c + c_1 \leq 4c$, then $T(3) + c_1 \leq 3c + c_1 \leq 4c$

True for any $c \geq c_1$

Proving the Hypothesis Inductively

We're starting to see a pattern…

Proving the Hypothesis Inductively

Approach: Assume our hypothesis is true for any $n' < n$;
Now prove it must also hold true for n.

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

$$T(n - 1) + c_1 \leq c \cdot n$$

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

$$T(n - 1) + c_1 \leq c \cdot n$$

By the inductive assumption, there is a c s.t. $T(n - 1) \leq (n - 1)c$

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

$$T(n - 1) + c_1 \leq c \cdot n$$

By the inductive assumption, there is a c s.t. $T(n - 1) \leq (n - 1)c$

So if we show that $(n - 1)c + c_1 \leq nc$, then…

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

$$T(n - 1) + c_1 \leq c \cdot n$$

By the inductive assumption, there is a c s.t. $T(n - 1) \leq (n - 1)c$

So if we show that $(n - 1)c + c_1 \leq nc$, then…

$$T(n - 1) + c_1 \leq (n - 1)c + c_1 \leq nc$$

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$$T(n) \leq c \cdot n$$

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By the inductive assumption, there is a c s.t. $T(n - 1) \leq (n - 1)c$

So if we show that $(n - 1)c + c_1 \leq nc$, then…

$$T(n - 1) + c_1 \leq (n - 1)c + c_1 \leq nc$$

True for any $c \geq c_1$

Proving the Hypothesis Inductively

Assume: There is a $c > 0$ s.t. $T(n - 1) \leq c \cdot (n - 1)$

Prove: There is a $c > 0$ s.t. $T(n) \leq c \cdot n$

$$T(n) \leq c \cdot n$$

$$T(n - 1) + c_1 \leq c \cdot n$$

By the inductive assumption, there is a c s.t. $T(n - 1) \leq (n - 1)c$

So if we show that $(n - 1)c + c_1 \leq nc$, then...

$$T(n - 1) + c_1 \leq (n - 1)c + c_1 \leq nc$$

True for any $c \geq c_1$

Therefore, we've proven our hypothesis for the Base Case, and inductively for the Recursive Case.
Therefore, the complexity of factorial is $\Theta(n)$

How much space is used?

```
factorial(n)
```

How much space is used?

`factorial(n-1)`
`factorial(n)`

How much space is used?

`factorial(n-2)`
`factorial(n-1)`
`factorial(n)`

How much space is used?

`factorial(n-3)`
`factorial(n-2)`
`factorial(n-1)`
`factorial(n)`

How much space is used?

⋮

`factorial(n-4)`
`factorial(n-3)`
`factorial(n-2)`
`factorial(n-1)`
`factorial(n)`

Tail Recursion

If the last thing we do in the function is a recursive call, we shouldn't need to create an entire stack of all the function calls…

```
def factorial(n: Int): Long =
    if(n <= 1){ 1 }
    else { n * factorial(n - 1) }
```

…smart compilers can often automatically convert to a loop…

```
def factorial(n: Int): Long = {
    var total = 1l
    for(i <- 1 until n){ total *= i }
    return total
}
```

Tail Recursion

The Scala compiler will attempt to turn tail recursion into a loop

If you add `@tailrec` before your function definition, the compiler will yell at you if it cannot do the conversion

Fibonacci

What about a function without tail recursion, or with multiple recursive calls?

What is the complexity of **`fibb(n)`**?

```
def fibb(n: Int): Long =
    if(n < 2){ 1 }
    else { fibb(n-1) + fibb(n-2) }
```

Next time...

Divide and Conquer

Recursion Trees