CSE 250
Data Structures

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QuickSort and Average Runtime
Textbook Ch. 15

Announcements

- WA1 due Wednesday at 11:59PM
- My office hours for today are cancelled, will hold them tomorrow instead

Recap - Merge Sort

Divide: Split the sequence in half

$D(n) = \Theta(n)$ (can do in $\Theta(1)$)

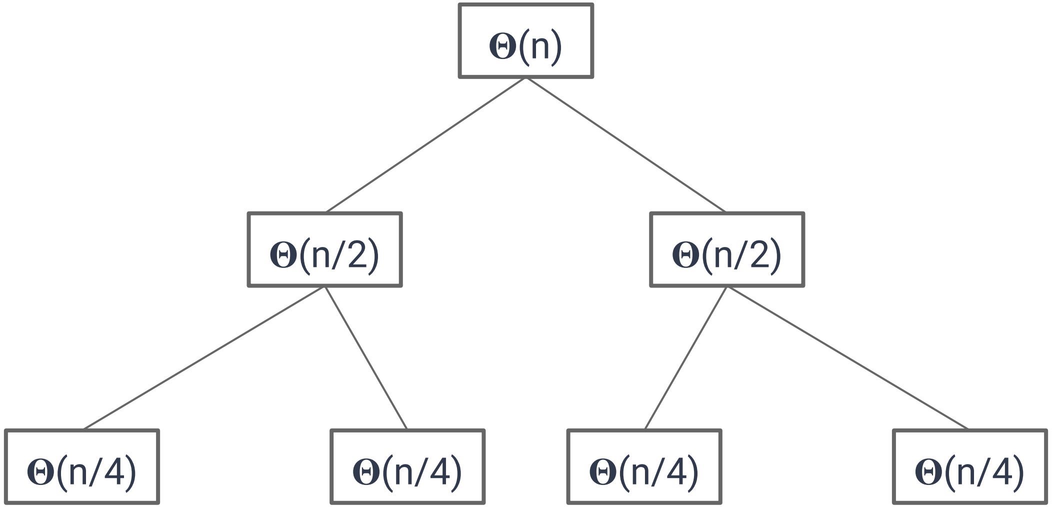
Conquer: Sort the left and right halves

$a = 2, b = 2, c = 1$

Combine: Merge halves together

$C(n) = \Theta(n)$

Merge Sort: Intuition



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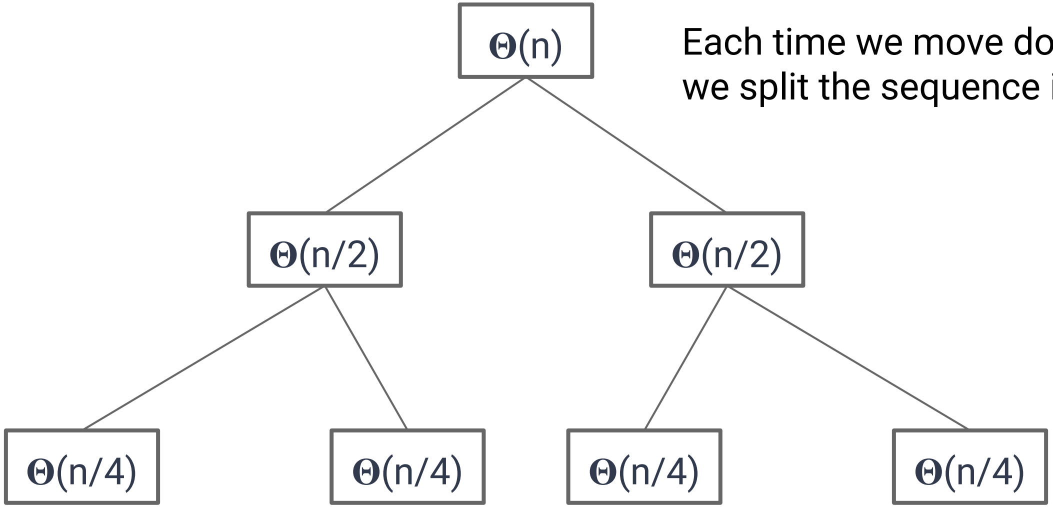

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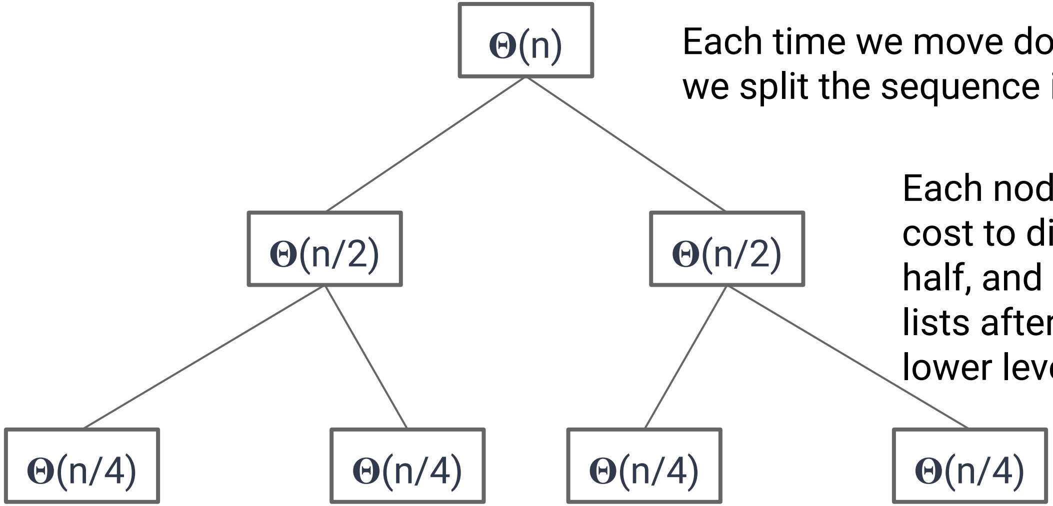

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Each node is labeled with the total cost to dividing the sequence in half, and combining the sorted lists after they are sorted by the lower levels

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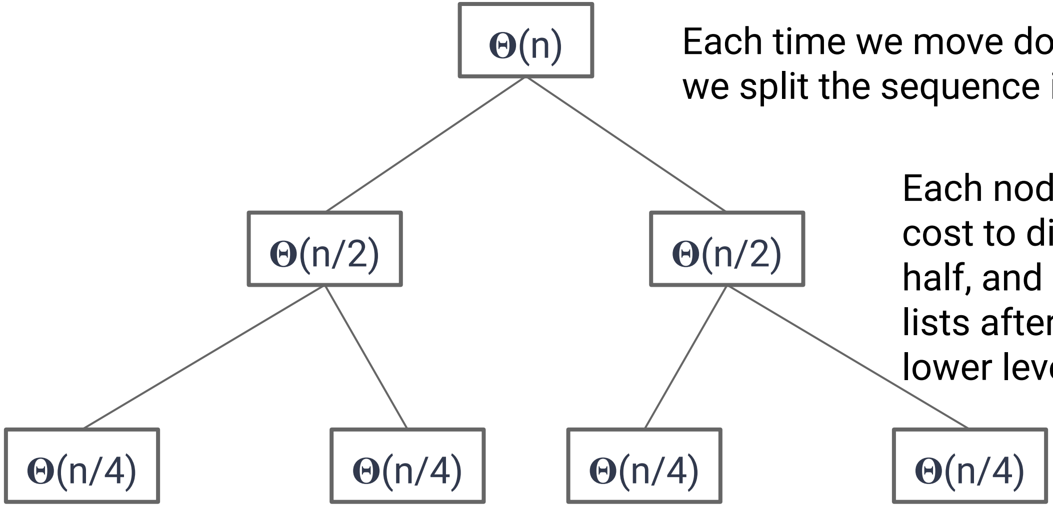

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Notice the total cost of each level is always $\Theta(n)$

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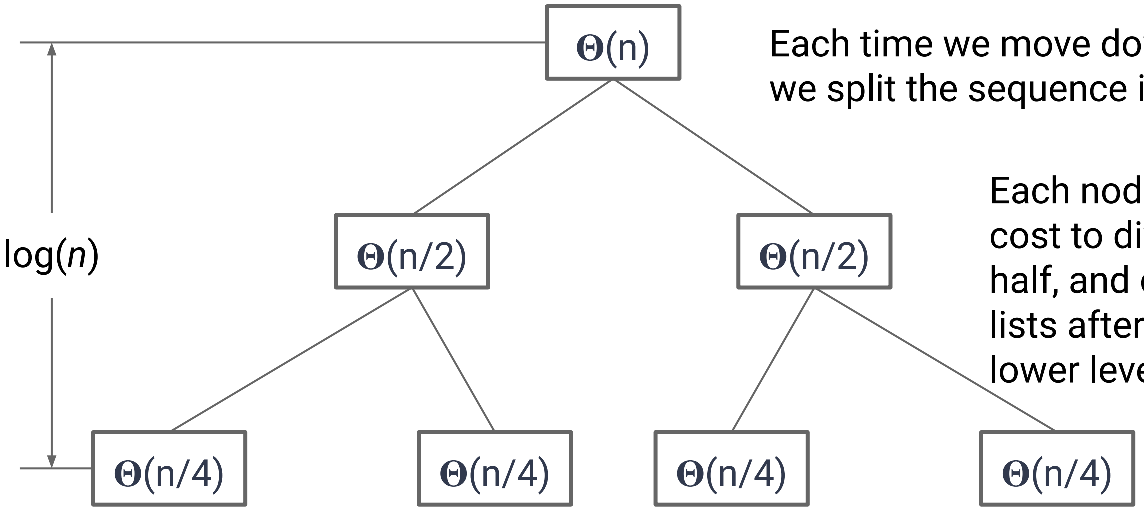
Because we divide in half at each level, we have log(n) levels



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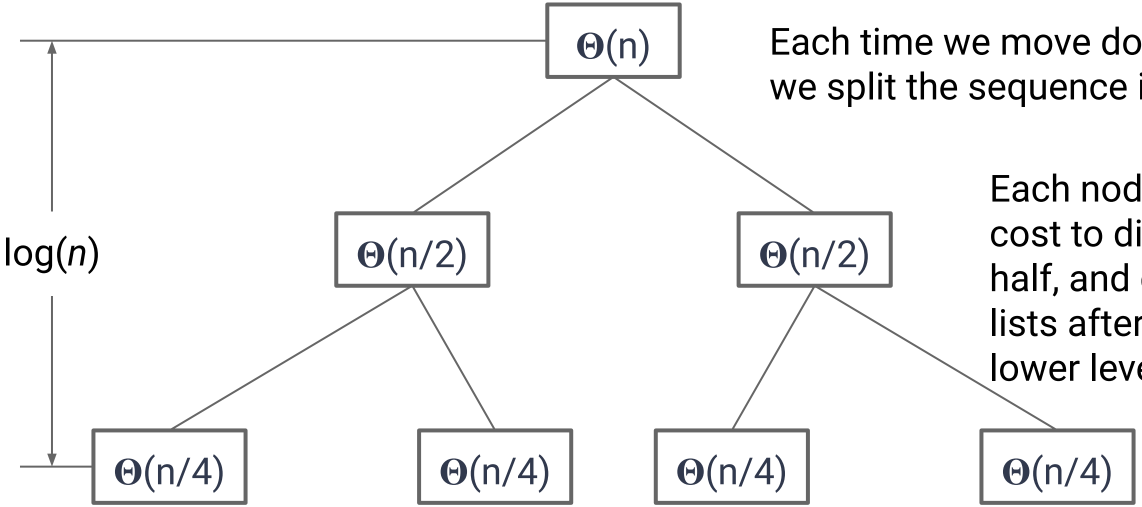

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Notice the total cost of each level is always $\Theta(n)$

Hypothesis: The cost of merge sort is n log(n)

Merge Sort: Proof by Induction

Base Case: $T(1) \leq c$

$c_0 \leq c$

True for any $c > c_0$

Merge Sort: Proof by Induction

Assume: $T(n/2) \leq c\ (n/2) \log(n/2)$

Show: $T(n) \leq cn \log(n)$

Merge Sort: Proof by Induction

Assume: $T(n/2) \leq c\,(n/2)\log(n/2)$

Show: $T(n) \leq cn\log(n)$

$$2 \cdot T(\frac{n}{2}) + c_1 + c_2 n \leq cn\log(n)$$

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By the assumption, and transitivity, we just need to show:

$$2c\frac{n}{2}\log\left(\frac{n}{2}\right) + c_1 + c_2 n \leq cn\log(n)$$

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Merge Sort: Proof by Induction

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$$\frac{c_1}{n \log(2)} + \frac{c_2}{\log(2)} \leq c$$

Merge Sort: Proof by Induction

$$c_1 + c_2 n \leq cn\log(2)$$

$$\frac{c_1}{n\log(2)} + \frac{c_2}{\log(2)} \leq c$$

Which is true for any

$$n_0 \geq \frac{c_1}{\log(2)} \quad \text{and} \quad c > \frac{c_2}{\log(2)} + 1$$

Merge Sort

Where is all of the "work" being done?

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The combine step

Merge Sort

Where is all of the "work" being done?

The combine step

Can we put the work in the divide step instead?

QuickSort

Idea: What if we divide our sequence around a particular value?

What value would we like to choose?

QuickSort

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What value would we like to choose? **Median**

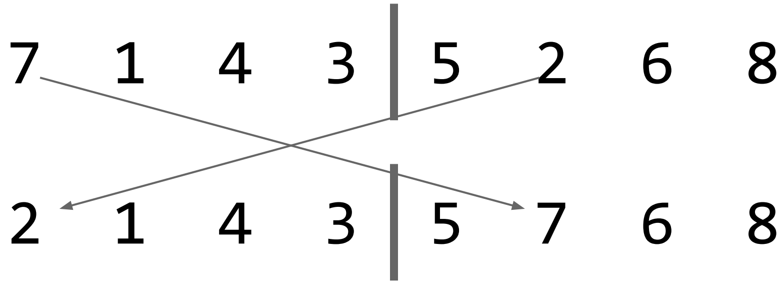
QuickSort: Idealized Version

7 1 4 3 5 2 6 8

QuickSort: Idealized Version

7 1 4 3 | 5 2 6 8

QuickSort: Idealized Version



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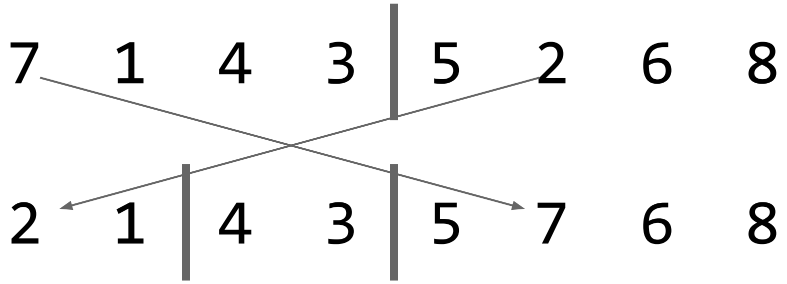

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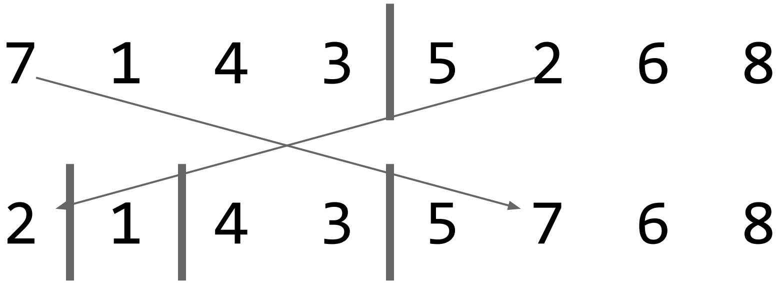

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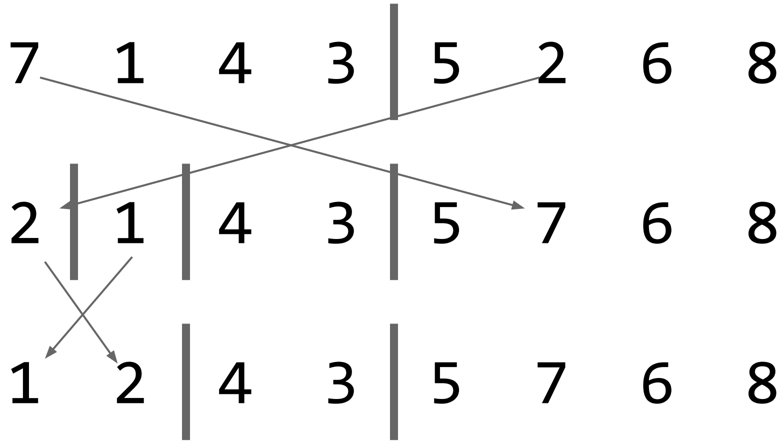

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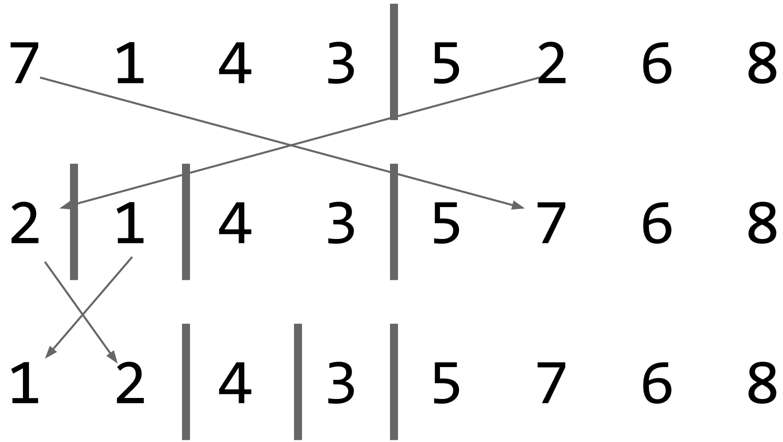

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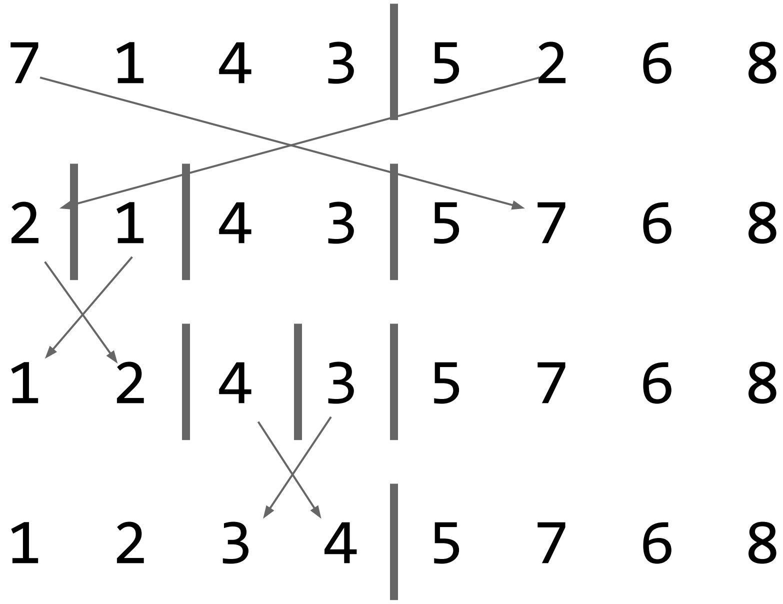

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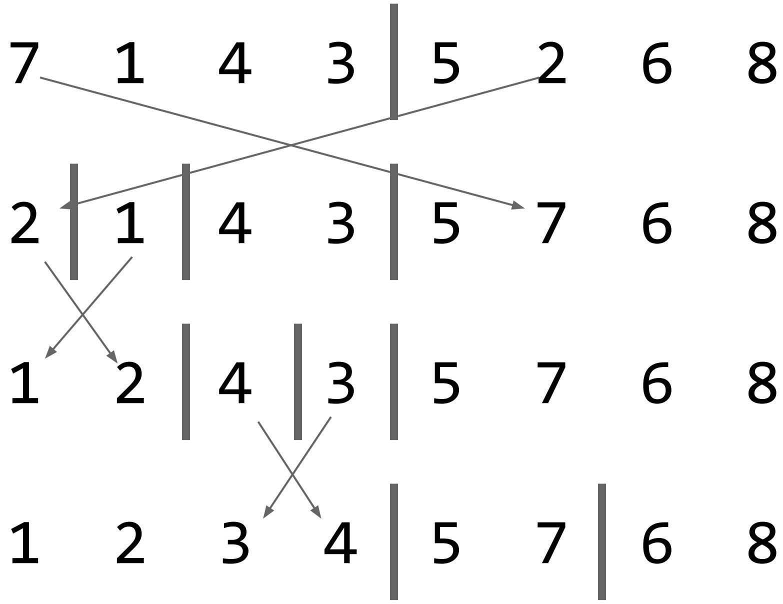

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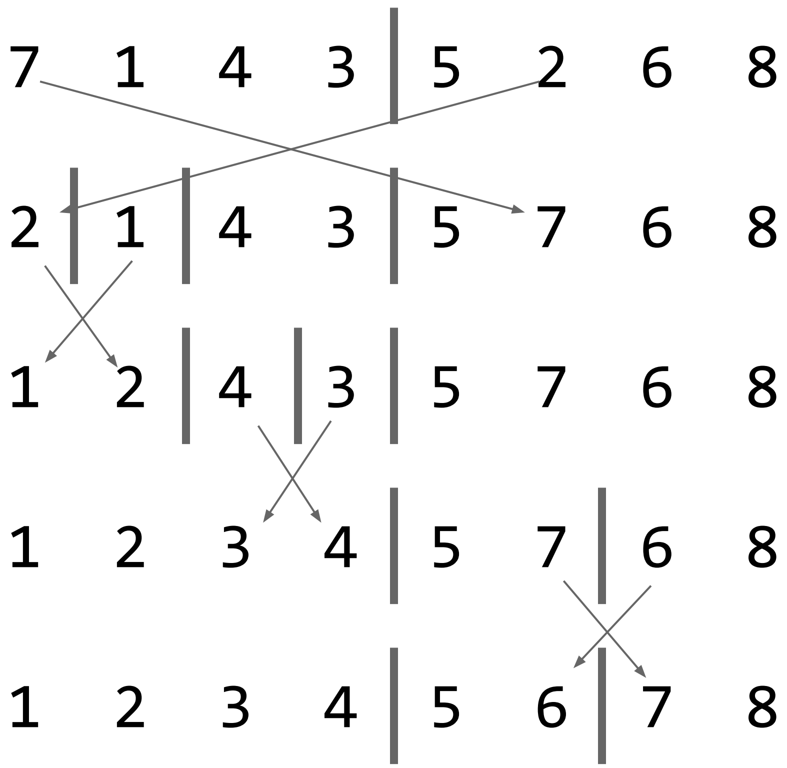

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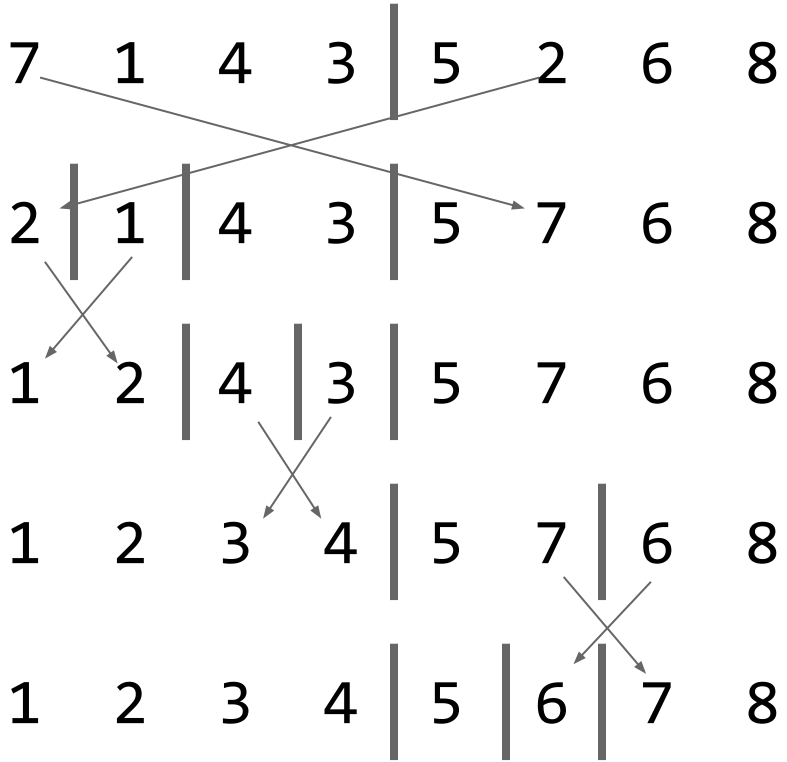

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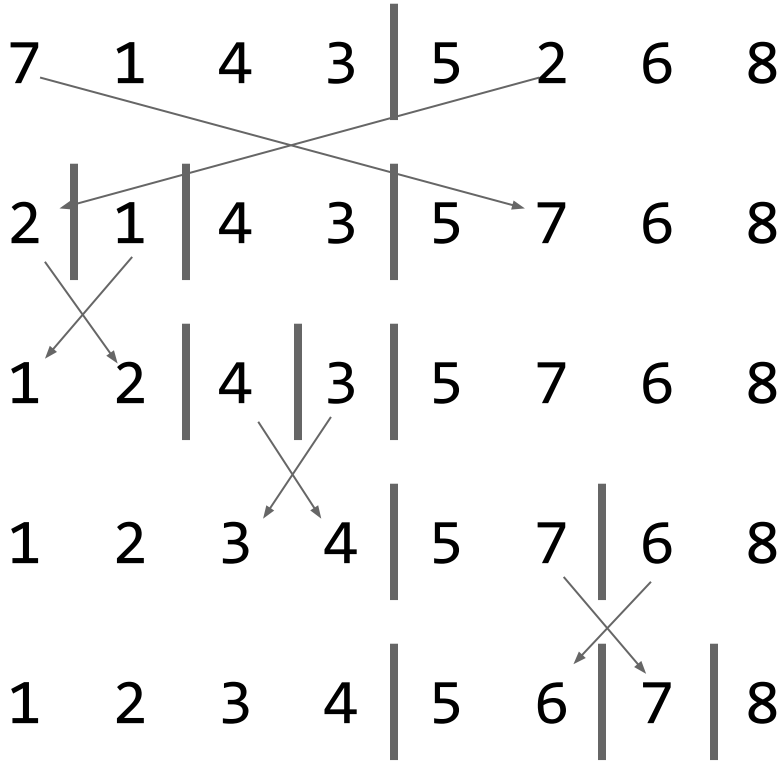

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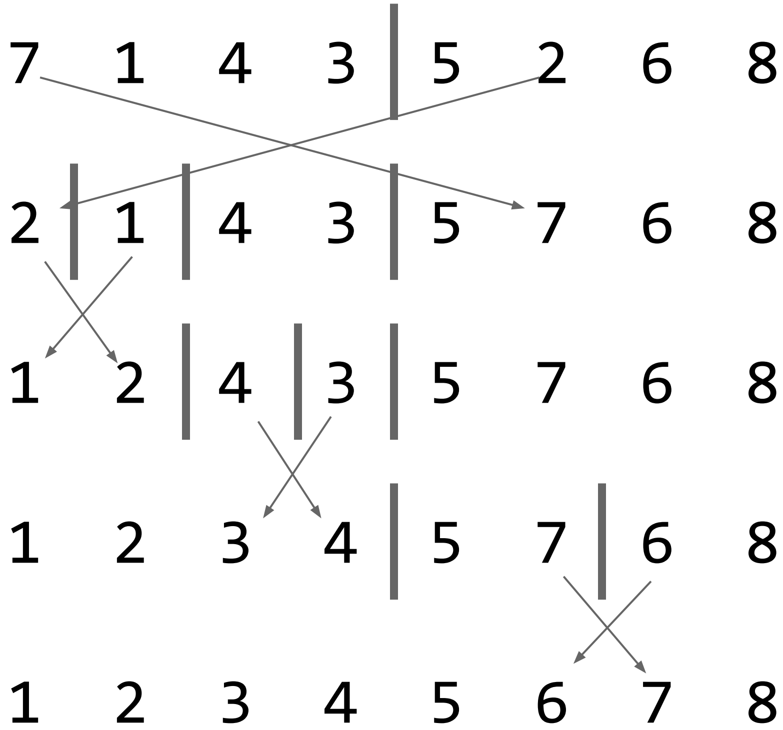

QuickSort: Idealized Algorithm

To sort an array of size *n*:

1. Pick a *pivot* value (median?)
2. Swap values until:
   a. elements at [1, *n*/2) are ≤ pivot
   b. elements at [*n*/2, *n*) are > pivot
3. Recursively sort the lower half
4. Recursively sort the upper half

QuickSort: Idealized Version

```
def idealizedQuickSort(arr: Array[Int], from: Int, until: Int): Unit = {
    if(until - from < 1) { return }
    val pivot = ???
    var low = from, high = until -1

    while(low < high) {
        while(arr(low) <= pivot && low < high){ low ++ }
        if(low < high) {
            while(arr(high) > pivot && low < high){ high -- }
            swap(arr, low, high)
        }
    }
    idealizedQuickSort(arr, from = 0,   until = low)
    idealizedQuickSort(arr, from = low, until = until)
}
```

Great! So…how do we find the median…?

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Finding the median takes $O(n \log(n))$ for an unsorted array :(

QuickSort: Hypothetical

Imagine a world where we can obtain a pivot in $O(1)$.

Now what is our complexity?

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Now what is our complexity?

$$T_{quicksort}(n) = \begin{cases} \Theta(1) & \textbf{if } n = 1 \\ 2 \cdot T(\frac{n}{2}) + \Theta(n) + 0 & \textbf{otherwise} \end{cases}$$

QuickSort: Hypothetical

Imagine a world where we can obtain a pivot in $O(1)$.
Now what is our complexity?

$$T_{quicksort}(n) = \begin{cases} \Theta(1) & \textbf{if } n = 1 \\ 2 \cdot T(\frac{n}{2}) + \Theta(n) + 0 & \textbf{otherwise} \end{cases}$$

Compare to Merge Sort:

$$T_{mergesort}(n) = \begin{cases} \Theta(1) & \textbf{if } n = 1 \\ 2 \cdot T(\frac{n}{2}) + \Theta(1) + \Theta(n) & \textbf{otherwise} \end{cases}$$

QuickSort: Attempt #2

So how can we pick a pivot value (in $O(1)$ time)?

QuickSort: Attempt #2

So how can we pick a pivot value (in O(1) time)?

Idea: Pick it randomly! On average, half the values will be lower.

QuickSort: Attempt #2

To sort an array of size *n*:

1. Pick a value at random as the *pivot*
2. Swap values until the array is subdivided into:
    a. *low*: array elements < *pivot*
    b. *pivot*
    c. *high:* array elements > pivot
3. Recursively sort *low*
4. Recursively sort *high*

QuickSort: Runtime

What is the worst-case runtime?

QuickSort: Worst-Case Scenario

What if we always pick the worst pivot?

`[8,7,6,5,4,3,2,1]`

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`[7,6,5,4,3,2,1],8,[]`

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```

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`[5,4,3,2,1],6,[],7,8`

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Remember: This is called the unqualified runtime…we don't take any extra context into account

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Is the worst case runtime representative?

No! (the actual runtime will almost always be faster)

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No! (the actual runtime will almost always be faster)

But what **can** we say about runtime?

QuickSort

Let's say we pick Xth largest element for our pivot.

What is the runtime ($T(n)$)?

QuickSort

Let's say we pick Xth largest element for our pivot.

What is the runtime ($T(n)$)?

$$\begin{cases} T(0) + T(n-1) + \Theta(n) & \textbf{if } X = 1 \\ T(1) + T(n-2) + \Theta(n) & \textbf{if } X = 2 \\ T(2) + T(n-3) + \Theta(n) & \textbf{if } X = 3 \\ .. & \\ T(n-2) + T(1) + \Theta(n) & \textbf{if } X = n-1 \\ T(n-1) + T(0) + \Theta(n) & \textbf{if } X = n \end{cases}$$

Probabilities

How likely are we to pick $X = k$ for any specific k?

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$$P[X = k] = 1/n$$

Probability Theory (Great Class...)

If I roll a d6 (6-sided die) *k* times,

what is the average roll over all possible outcomes?

k = 1

If I rolled a d6 1 time…

Roll	Probability	Outcome
⚀	1/6	1
⚁	1/6	2
⚂	1/6	3
⚃	1/6	4
⚄	1/6	5
⚅	1/6	6

Expected Value

The **Expected Value** of a random variable (ie the number rolled on the d6) is the sum of all outcomes times the probability of that outcome

$$\sum_i Probability_i \cdot Contribution_i$$

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$$\sum_{i=1}^{6} \frac{1}{6} i = \frac{1}{6} \cdot 1 \frac{1}{6} \cdot 2 \frac{1}{6} \cdot 3 \frac{1}{6} \cdot 4 \frac{1}{6} \cdot 5 \frac{1}{6} \cdot 6 = 3.5$$

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We refer to the expected value of a random variable as ***E[X]***

Independent Events

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If X and Y are our dice rolls, then $E[X+Y] = E[X] + E[Y] = 3.5 + 3.5 = 7$

QuickSort Runtime

Now we can write our runtime function in terms of random variables:

$$T(n) = \begin{cases} \Theta(1) & \textbf{if } n \leq 1 \\ T(0) + T(n-1) + \Theta(n) & \textbf{if } n > 1 \wedge X = 1 \\ T(1) + T(n-2) + \Theta(n) & \textbf{if } n > 1 \wedge X = 2 \\ T(2) + T(n-3) + \Theta(n) & \textbf{if } n > 1 \wedge X = 3 \\ .. & \\ T(n-2) + T(1) + \Theta(n) & \textbf{if } n > 1 \wedge X = n-1 \\ T(n-1) + T(0) + \Theta(n) & \textbf{if } n > 1 \wedge X = n \end{cases}$$

QuickSort Runtime

…and convert it to the expected runtime over the variable ***X***

$$E[T(n)] = \begin{cases} \Theta(1) & \textbf{if } n \leq 1 \\ E[T(X-1) + T(n-X)] + \Theta(n) & \textbf{otherwise} \end{cases}$$

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This looks like the runtime of MergeSort, so now our hypothesis is that our **Expected Runtime** is ***n* log(*n*)**

Back to Induction

Hypothesis: $E[T(n)] \in O(n \log(n))$

Base Case

Base Case: $E[T(1)] \leq c\ (1 \log(1))$

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$E[T(1)] \leq c\ (1 \cdot 0)$

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$$E[T(1)] \leq c\ (1 \cdot 0)$$

$$E[T(1)] \nleq 0$$

Base Case (Take 2)

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$$T(0) + T(1) + 2c_1 \leq 2c$$

$$2c_0 + 2c_1 \leq 2c$$

True for any $c \geq c_0 + c_1$

Inductive Case

Assume: $E[T(n')] \leq c\ (n' \log(n'))$ for **all** $n' < n$

Show: $E[T(n)] \leq c\ (n \log(n))$

Inductive Case

Assume: *E*[*T*(*n'*)] ≤ *c* (*n'* log(*n'*)) for **all** *n'* < *n*

Show: *E*[*T*(*n*)] ≤ *c* (*n* log(*n*))

$$\frac{2}{n}\left(\sum_{i=0}^{n-1} E[T(i)]\right) + c_1 \leq cn\log(n)$$

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$$c\frac{2\log(n)}{n}\left(\frac{(n-1)(n-1+1)}{2}\right) + c_1 \leq cn\log(n)$$

$$c\frac{\log(n)}{n}\left(n^2 - n\right) + c_1 \leq cn\log(n)$$

Inductive Case

$$c\frac{2}{n}\left(\sum_{i=0}^{n-1} i\log(n)\right) + c_1 \leq cn\log(n)$$

$$c\frac{2\log(n)}{n}\left(\sum_{i=0}^{n-1} i\right) + c_1 \leq cn\log(n)$$

$$c\frac{2\log(n)}{n}\left(\frac{(n-1)(n-1+1)}{2}\right) + c_1 \leq cn\log(n)$$

$$c\frac{\log(n)}{n}\left(n^2 - n\right) + c_1 \leq cn\log(n)$$

$$cn\log(n) - c\log(n) + c_1 \leq cn\log(n)$$

Inductive Case

$$c\frac{2}{n}\left(\sum_{i=0}^{n-1} i\log(n)\right) + c_1 \leq cn\log(n)$$

$$c\frac{2\log(n)}{n}\left(\sum_{i=0}^{n-1} i\right) + c_1 \leq cn\log(n)$$

$$c\frac{2\log(n)}{n}\left(\frac{(n-1)(n-1+1)}{2}\right) + c_1 \leq cn\log(n)$$

$$c\frac{\log(n)}{n}\left(n^2 - n\right) + c_1 \leq cn\log(n)$$

$$cn\log(n) - c\log(n) + c_1 \leq cn\log(n)$$

$$c_1 \leq c\log(n)$$

QuickSort

So…is QuickSort $O(n \log(n))$...?

No!

What guarantees do you get?

If $f(n)$ is a Tight Bound

The algorithm always runs in $cf(n)$ steps

If $f(n)$ is a Worst-Case Bound

The algorithm always runs in at most $cf(n)$

If $f(n)$ is an Amortized Worst-Case Bound

n invocations of the algorithm **always** run in $cnf(n)$ steps

If $f(n)$ is an Average Bound

…we don't have any guarantees

What guarantees do you get?

If *f*(*n*) is a Tight Bound
The algorithm always runs in $cf(n)$ steps

← Unqualified runtime

If *f*(*n*) is a Worst-Case Bound
The algorithm always runs in at most $cf(n)$

If *f*(*n*) is an Amortized Worst-Case Bound
n invocations of the algorithm **always** run in $cnf(n)$ steps

If *f*(*n*) is an Average Bound
…we don't have any guarantees