CSE 250

Data Structures

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AVL Trees

Announcements

- PA3 Tests due tonight @11:59PM
- PA3 Implementation due Sunday @ 11:59PM
    - Recitation this week will have some PA3 related content again

BST Operations

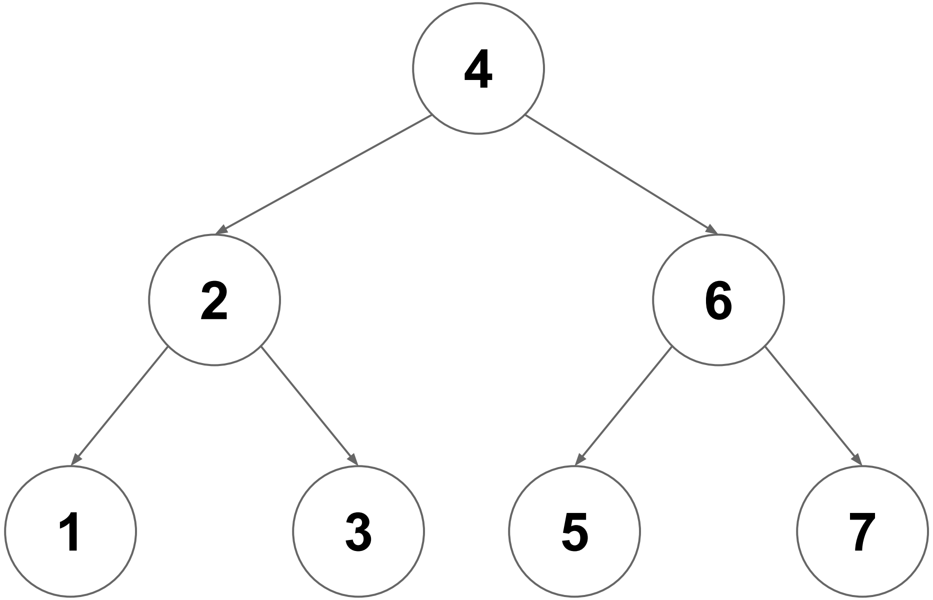
Operation	Runtime
`find`	$O(d)$
`insert`	$O(d)$
`remove`	$O(d)$

What is the runtime in terms of n? $O(n)$

$\log(n) \leq d \leq n$

Tree Depth vs Size

If height(left) ≈ height(right)



$d = O(\log(n))$

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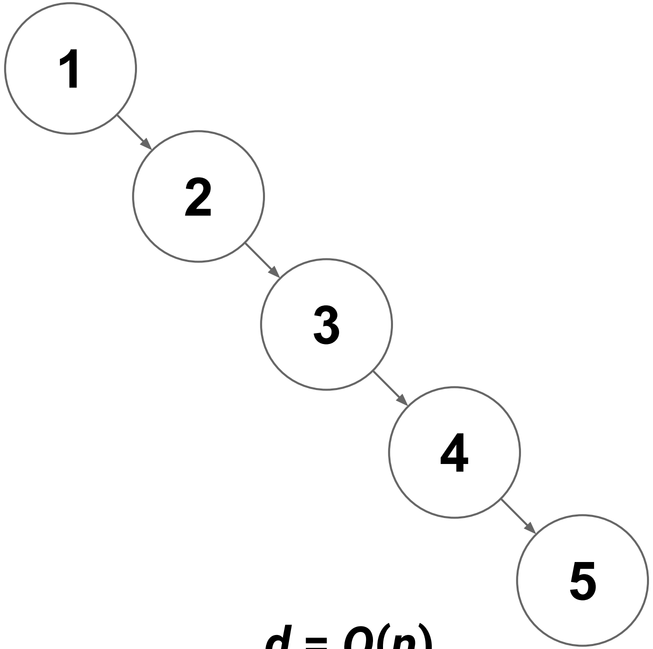

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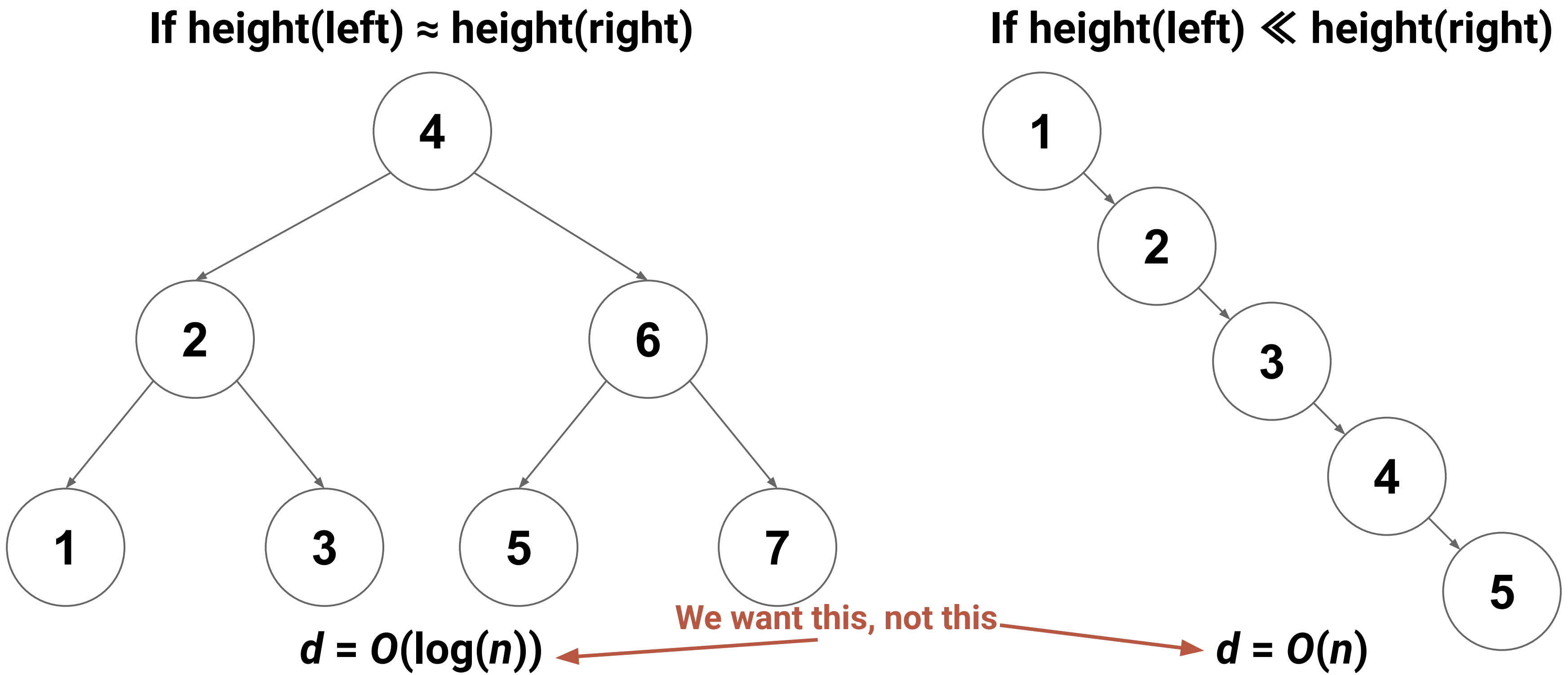

Keeping Depth Small - Two Approaches

Option 1

Keep tree **balanced**: subtrees **+/-1** of each other in height

(add a field to track amount of "imbalance")

Option 2

Keep leaves at some minimum depth (***d*/2**)

(Add a color to each node marking it as "red" or "black")

Balanced Trees

Balanced Trees are good: Faster `find`, `insert`, `remove`

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What do we mean by balanced? **$|\text{height(left)} - \text{height(right)}| \leq 1$**

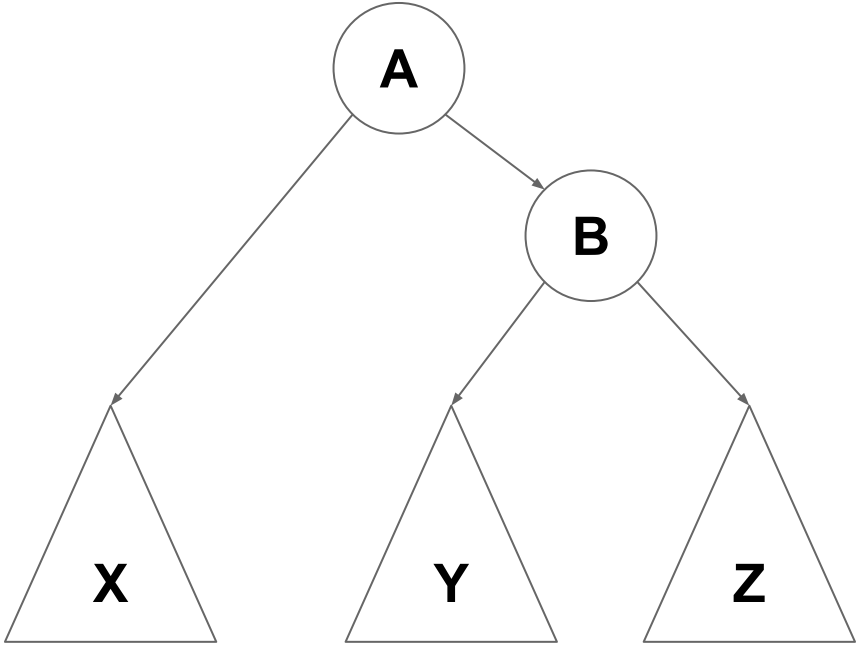
Balanced Trees

Balanced Trees are good: Faster `find`, `insert`, `remove`

What do we mean by balanced? **|height(left) - height(right)| ≤ 1**

How do we keep a tree balanced?

Rebalancing Trees (rotations)



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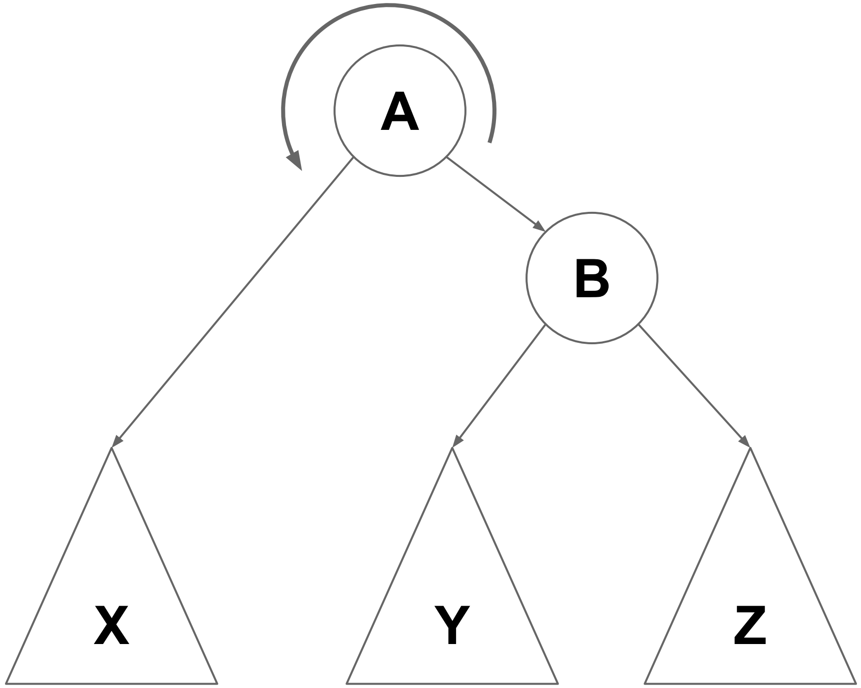
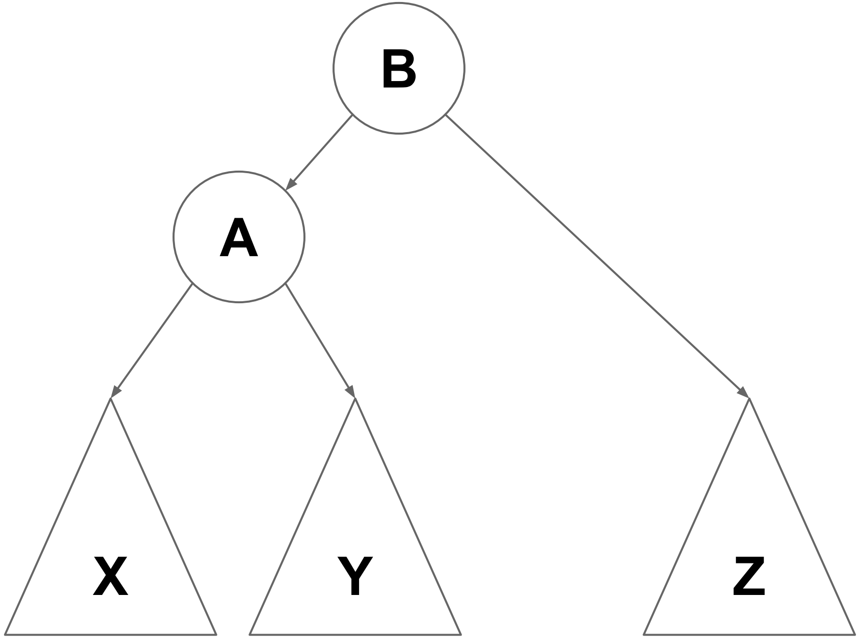


Rotate(A, B)

Rebalancing Trees (rotations)



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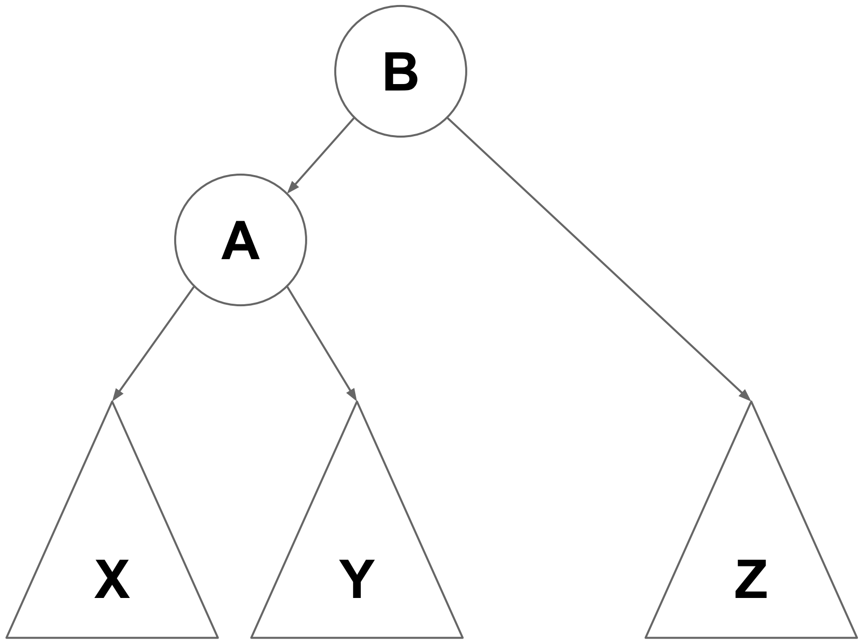
Rebalancing Trees (rotations)

A became **B**'s left child

B's left child became **A**'s right child



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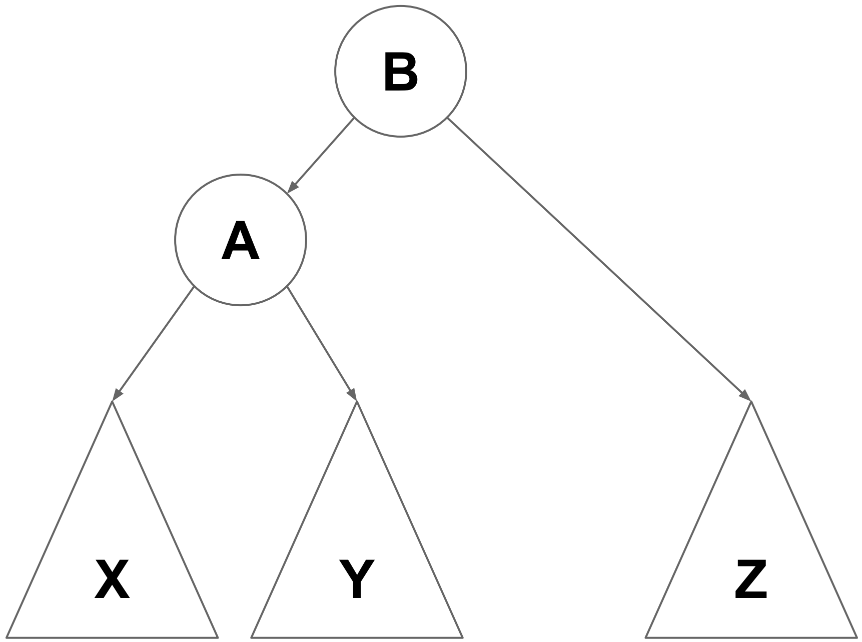
Is ordering maintained?



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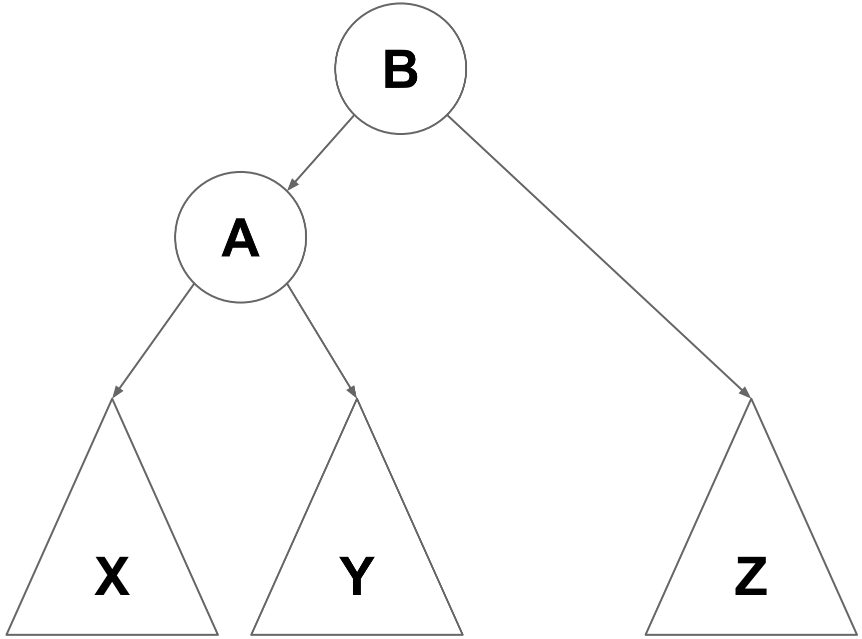

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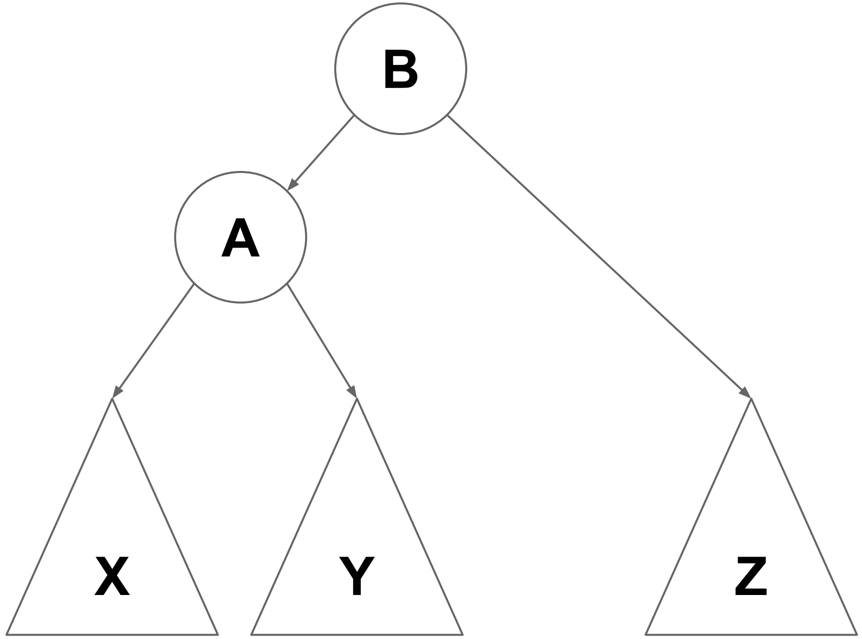

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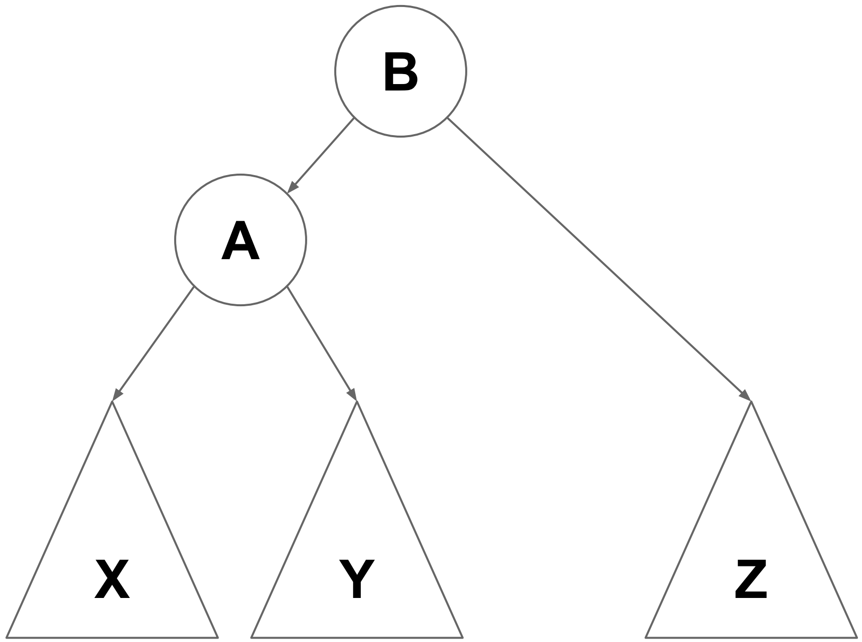

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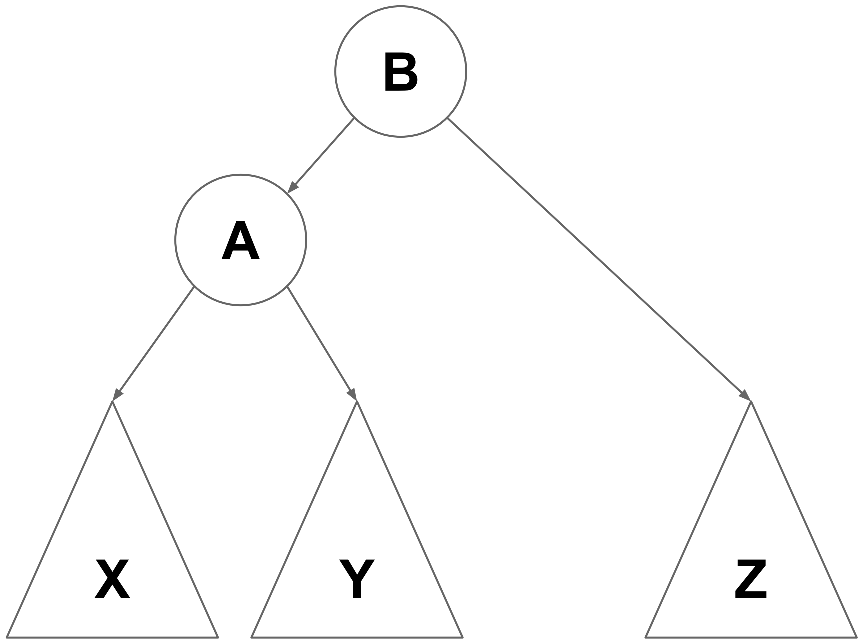
This is called a left rotation

(right rotation is the opposite)



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How does a rotation affect height?

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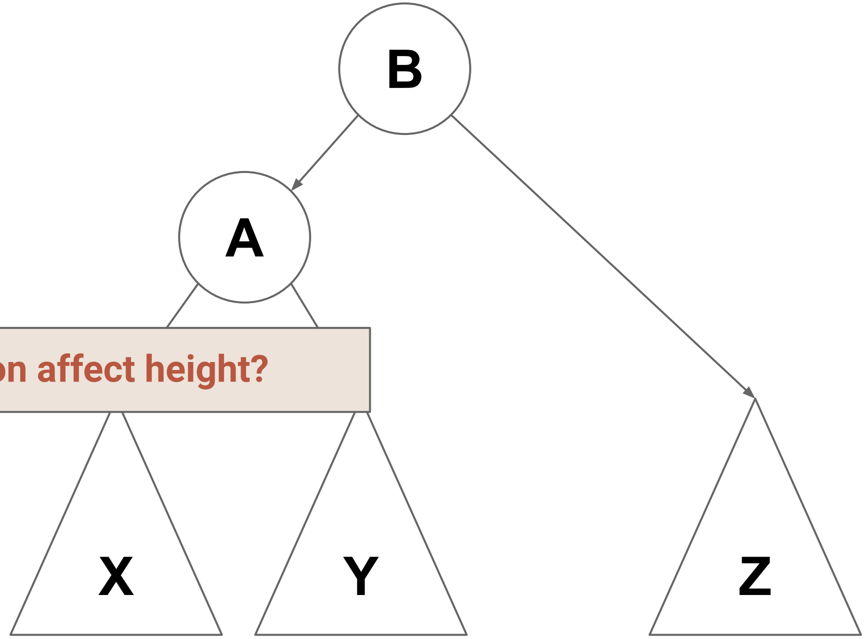
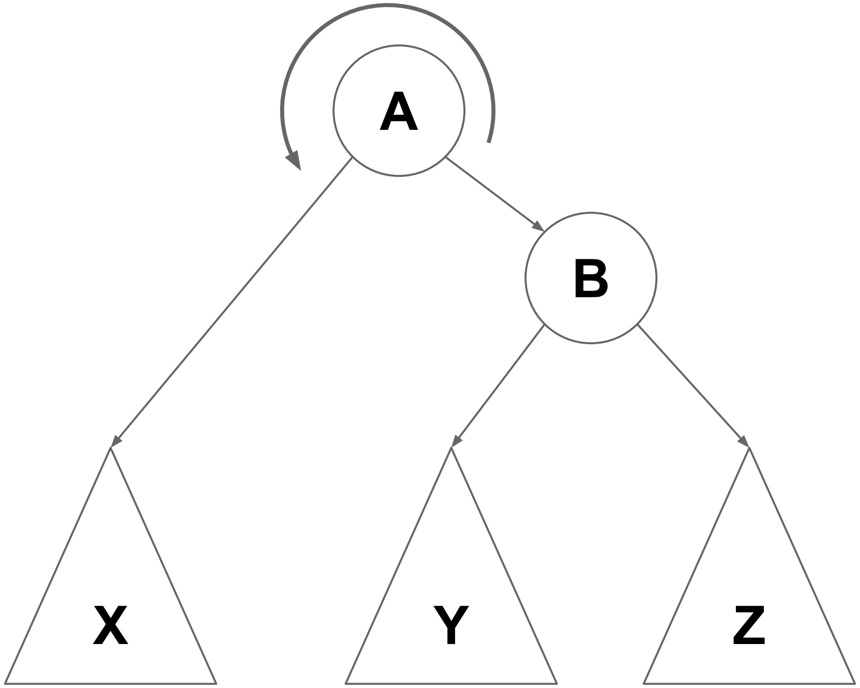


Rotate(A, B)

Rebalancing Trees (rotations)

Before Rotation:

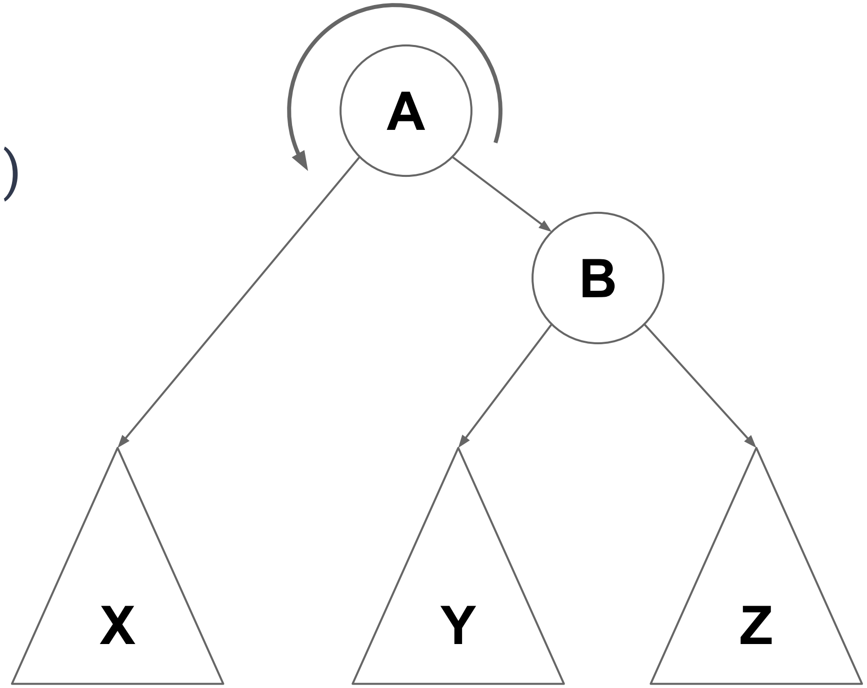


Rotate(A, B)

Rebalancing Trees (rotations)

Before Rotation:

$h(A) = 1 + \max(h(X), 1 + \max(h(Y), h(Z)))$



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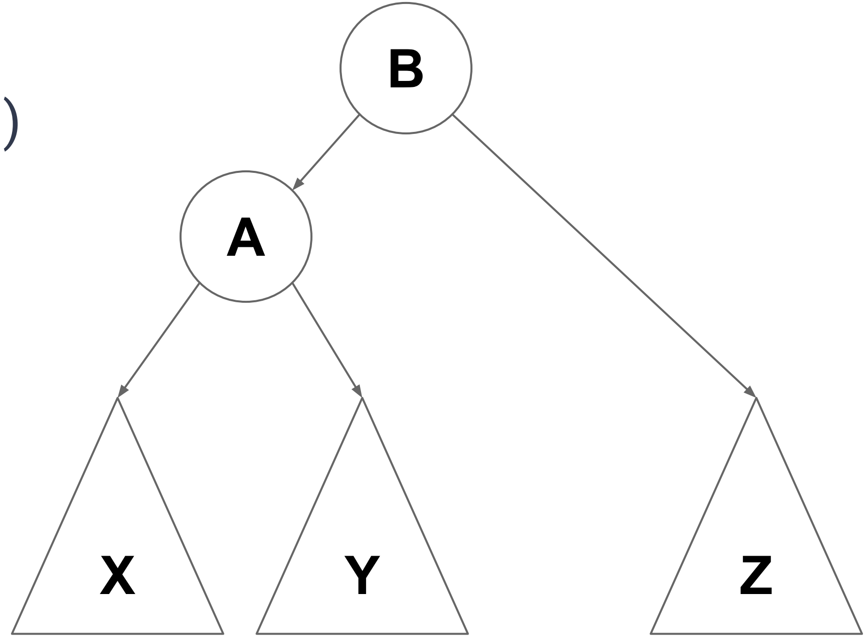

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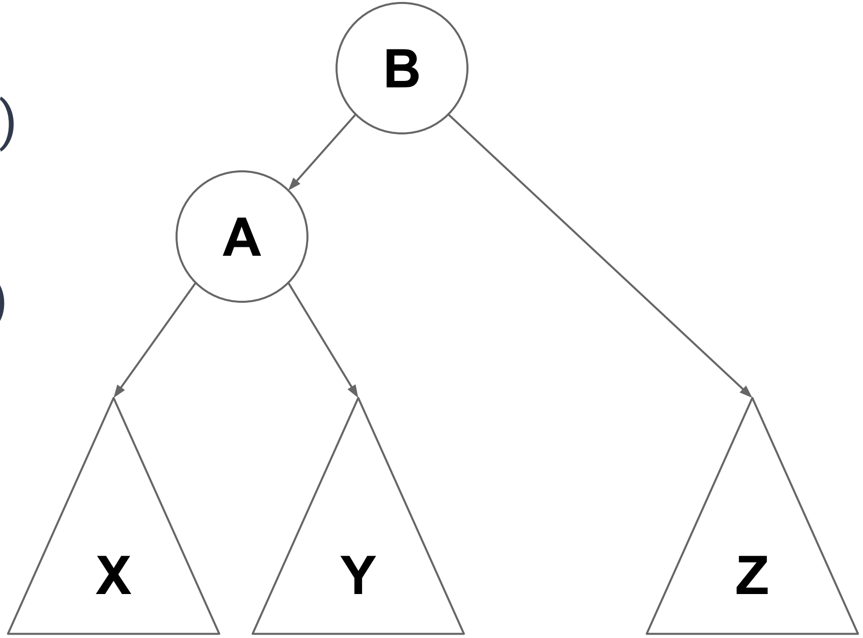


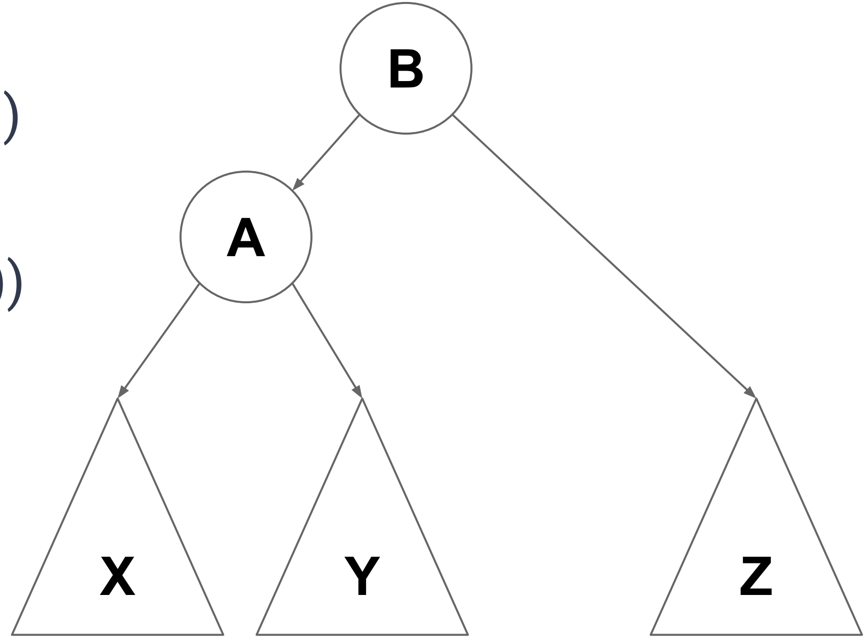
Rotate(A, B)

Rebalancing Trees (rotations)

Before Rotation:
h(A) = 1 + max(h(X), 1 + max(h(Y), h(Z))

After Rotation:
h(B) = 1 + max(1 + max(h(X),h(Y)), h(Z))

- If **X** was the tallest of **X,Y,Z** our total height increased by 1.
- If **Z** was the tallest our total height decreased by 1.
- If **X,Z** same height, or **Y** is the tallest then total is unchanged



Rotate(A, B)

AVL Trees

AVL Trees

An **AVL tree** (**A**delson-**V**elsky and **L**andis) is a ***BST*** where every subtree is depth-balanced

Remember: Tree depth = height(root)

Balanced: |height(root.left) - height(root.right)| ≤ 1

AVL Trees

Define $\texttt{balance}(v) = \texttt{height}(v.\mathit{right}) - \texttt{height}(v.\mathit{left})$

Goal: Maintaining $\texttt{balance}(v) \in \{-1, 0, 1\}$

- $\texttt{balance}(v) = 0$ → "***v*** is balanced"
- $\texttt{balance}(v) = -1$ → "***v*** is left-heavy"
- $\texttt{balance}(v) = 1$ → "***v*** is right-heavy"

AVL Trees

Define **balance(*v*) = height(*v.right*) - height(*v.left*)**

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What does enforcing this gain us?

AVL Trees - Depth Bounds

Question: Does the AVL property result in any guarantees about depth?

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YES! Depth balance forces a maximum possible depth of **log(*n*)**

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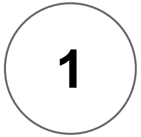
YES! Depth balance forces a maximum possible depth of **log(*n*)**

Proof Idea: An AVL tree with depth ***d*** has "enough" nodes

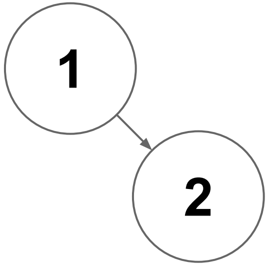
AVL Trees - Depth Bounds

Let **`minNodes(`*`d`*`)`** be the min number of nodes an in AVL tree of depth ***d***

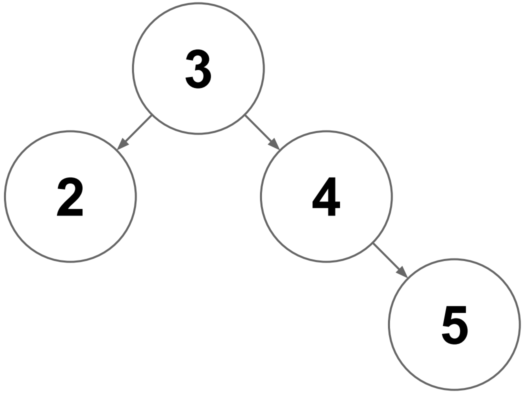
minNodes(0) = 1



minNodes(1) = 2

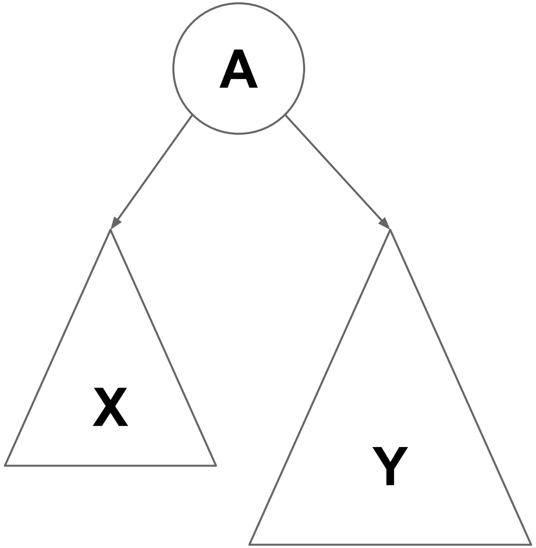


minNodes(2) = 4



AVL Trees - Depth Bounds

For any tree of depth ***d***:



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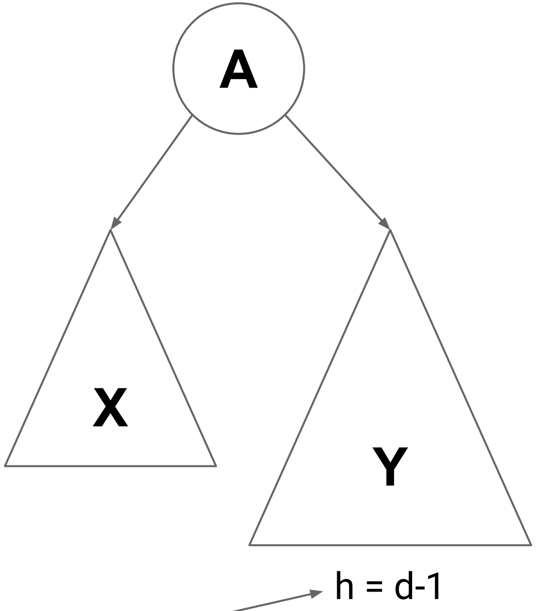
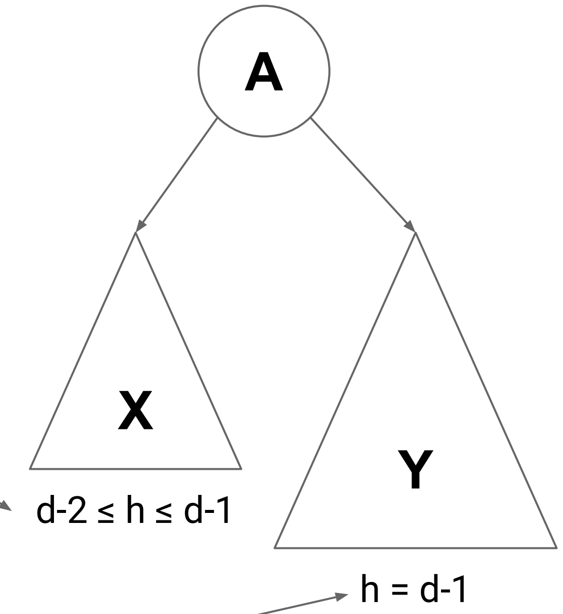


At least one subtree must have depth of ***d* - 1**
(because total depth is ***d***)

AVL Trees - Depth Bounds

For any tree of depth ***d***:

The other subtree must have a depth of at least ***d* - 2** because the AVL constraint does not allow it to differ by more than 1



At least one subtree must have depth of ***d* - 1** (because total depth is ***d***)

AVL Tree - Depth Bounds

For $\boldsymbol{d} < 1$:

minNodes($\boldsymbol{d}$) = 1 + minNodes($\boldsymbol{d}$ **- 1**) + minNodes($\boldsymbol{d}$ **- 2**)

AVL Tree - Depth Bounds

For $\boldsymbol{d} < 1$:

$\text{minNodes}(\boldsymbol{d}) = 1 + \boxed{\text{minNodes}(\boldsymbol{d - 1}) + \text{minNodes}(\boldsymbol{d - 2})}$

This is the Fibonacci Sequence!

AVL Tree - Depth Bounds

For $d < 1$:

minNodes(d) = 1 + $\boxed{\text{minNodes}(d - 1) + \text{minNodes}(d - 2)}$

This is the Fibonacci Sequence!

What is the d^{th} term of the Fibonacci sequence?

Coarse approximation: minNodes(d) = $\Omega(1.5^d)$

https://en.wikipedia.org/wiki/Fibonacci_sequence

AVL Tree - Depth Bounds

minNodes($\boldsymbol{d}$) = $\Omega(1.5^{d})$

AVL Tree - Depth Bounds

minNodes(d) = $\Omega(1.5^d)$

$$n \geq c1.5^d$$

AVL Tree - Depth Bounds

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AVL Tree - Depth Bounds

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AVL Tree - Depth Bounds

minNodes(***d***) = **Ω**(1.5^d)

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$$\log_2\left(\frac{n}{c}\right) \geq d\log_2(1.5)$$

$$\frac{\log_2\left(\frac{n}{c}\right)}{\log_2(1.5)} \geq d$$

AVL Tree - Depth Bounds

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$$\frac{\log_2(n))}{\log_2(1.5)} - \frac{\log_2(c)}{\log_2(1.5)} \geq d$$

AVL Tree - Depth Bounds

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All constants

AVL Tree - Depth Bounds

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$$d \in O(\log_2(n))$$

AVL Tree - Depth Bounds

minNodes(***d***) = Ω(1.5^{d})

$$n \geq c1.5^d$$

$$\log_2\left(\frac{n}{c}\right) \geq \log_2 \ldots$$

$$\log_2\left(\frac{n}{c}\right) \geq d\log_2(1.5)$$

$$\log_2\left(\frac{n}{c}\right) \ldots \geq d$$

$$\frac{\ldots}{\log_2(1.5)} - \frac{\ldots g_2(c)}{\log_2(1.5)} \geq d$$

$$d \in O(\log_2(n))$$

Therefore if we enforce the AVL constraint, then a tree with *n* nodes will have logarithmic depth

AVL Tree - Depth Bounds

minNodes(***d***) = $\Omega(1.5^d)$

$$n \geq c1.5^d$$

$$\log_2\left(\frac{n}{c}\right) \geq d$$

$$\log_2\left(\frac{n}{c}\right) \geq \log$$

Therefore if we enforce the AVL constraint, then a tree with *n* nodes will have logarithmic depth

So how do we enforce the constraint?

$$\frac{\ldots g_2(c)}{\ldots_2(1.5)} \geq d$$

$$\log_2\left(\frac{n}{c}\right) \geq d\log_2(1.5)$$

$$d \in O(\log_2(n))$$

Enforcing the AVL Constraint

- Computing `balance()` on the fly is expensive
  - `balance()` calls `height()` twice
  - Computing `height()` requires visiting every node

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Idea: Store height of each node at the node

Better Idea: Just store the balance factor (only needs 2 bits)

Enforcing the AVL Constraint

```
class AVLNode[K,V](
  var key: K,
  var value: V,
  var parent: Option[AVLNode[K,V]],
  var left: Tree[K,V],
  var right: Tree[K,V],
  var isLeftHeavy : Boolean, // true if height(right) - height(left) == -1
  var isRightHeavy : Boolean, // true if height(right) - height(left) == 1
)
```

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```

Add fields to track balance, and update them during insertion/removal

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Assume we have a valid AVL tree and we modify it. How might this break our AVL constraint?

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- What is the effect on the height of `insert`? Increases by **at most** 1
- What is the effect on the height of `remove`?

Enforcing the AVL Constraint

Assume we have a valid AVL tree and we modify it. How might this break our AVL constraint?

- What is the effect on the height of `insert`? Increases by **at most** 1
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Enforcing the AVL Constraint

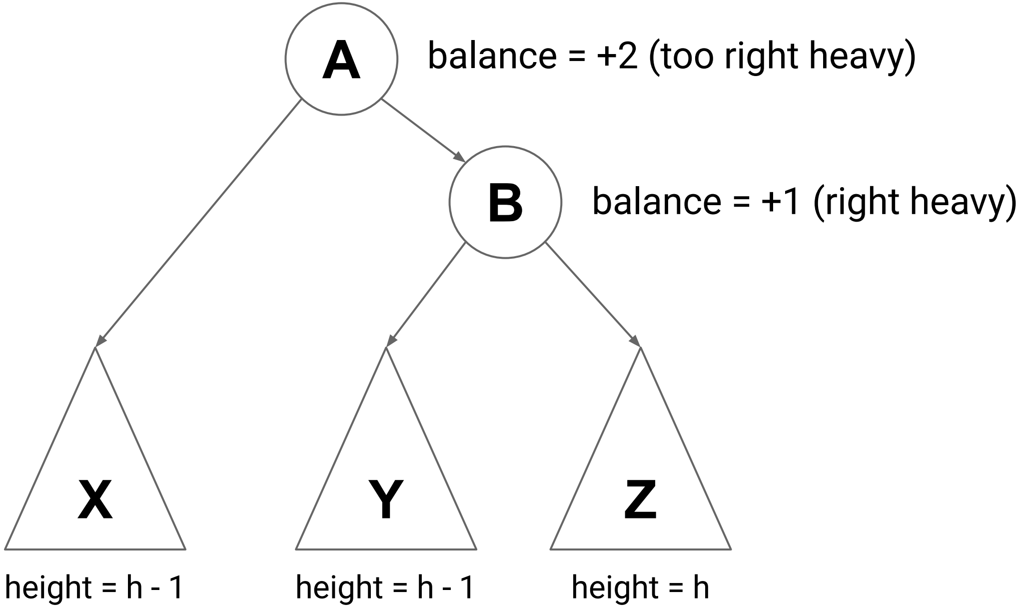
Assume we have a valid AVL tree and we modify it. How might this break our AVL constraint?

- What is the effect on the height of `insert`? Increases by **at most** 1
- What is the effect on the height of `remove`? Decreases by **at most** 1

Therefore after an operation that modifies an AVL tree, the difference in heights can be **at most** 2.

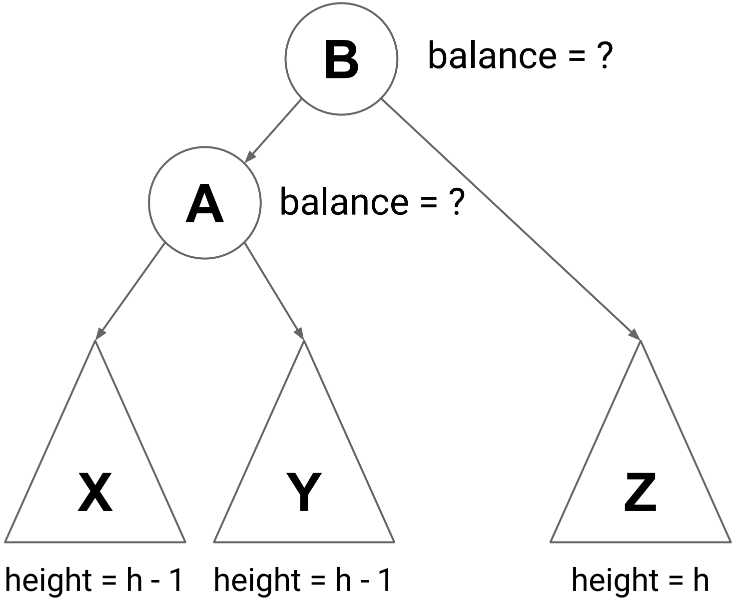
What are the exact ways this broken constraint might show up?

Enforcing the AVL Constraint: Case 1



How can we fix this?

Enforcing the AVL Constraint: Case 1



How can we fix this? **`rotate(A,B)`**

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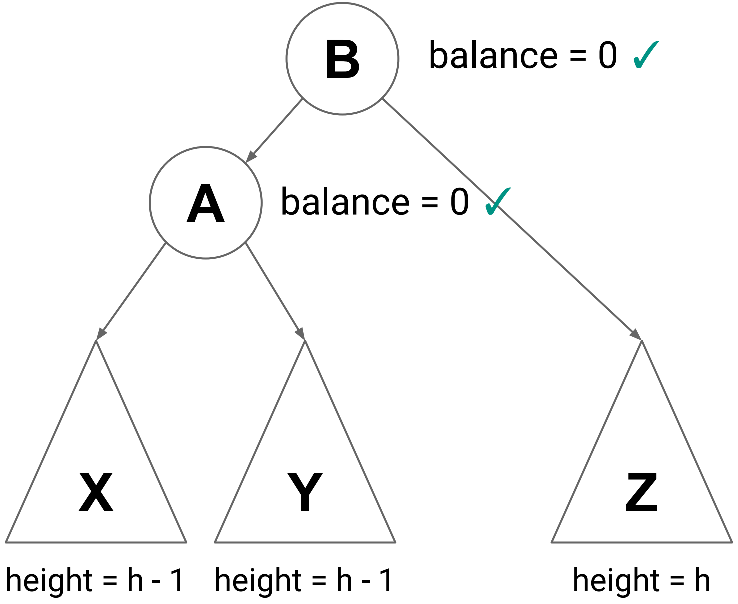

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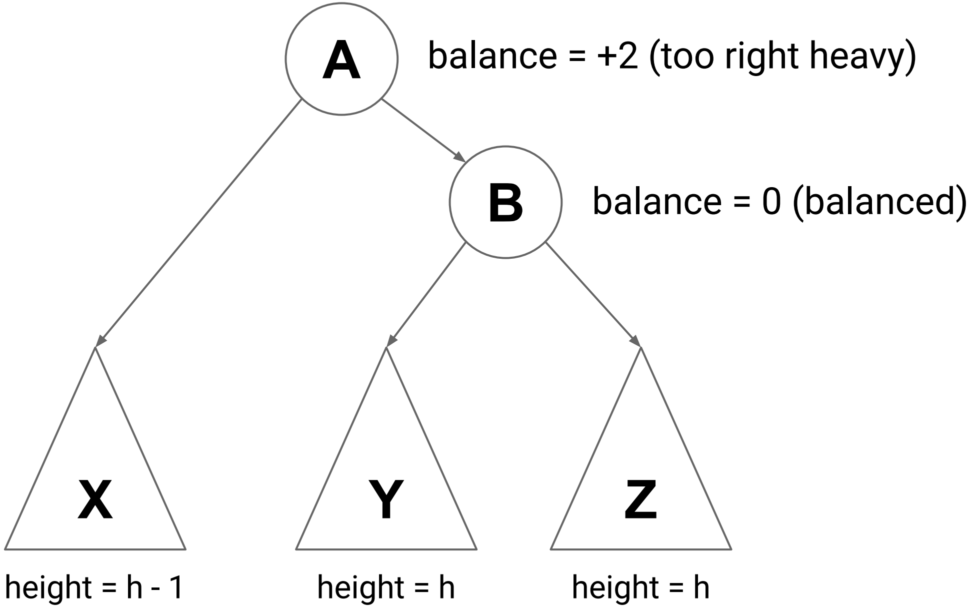
Enforcing the AVL Constraint: Case 2



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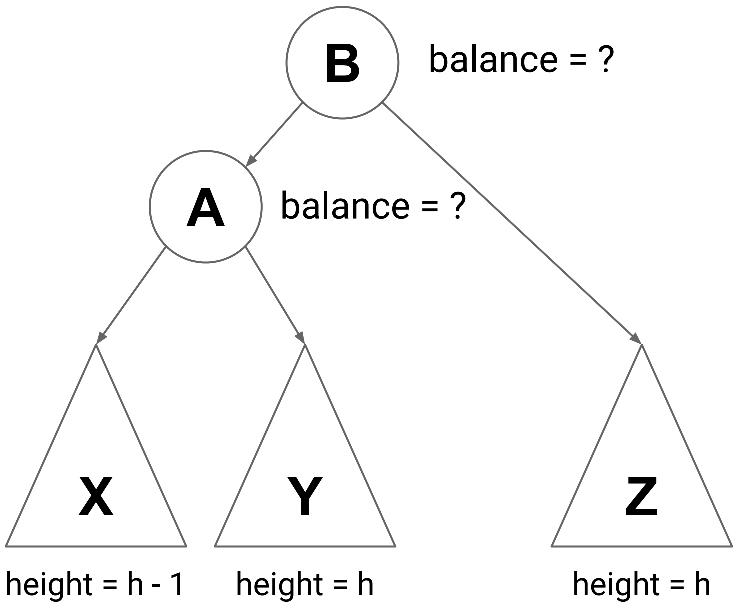
Enforcing the AVL Constraint: Case 2



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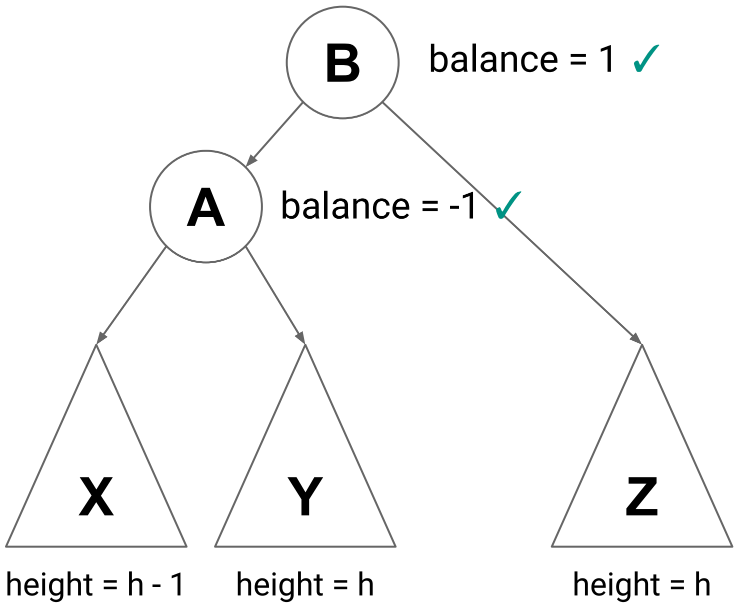

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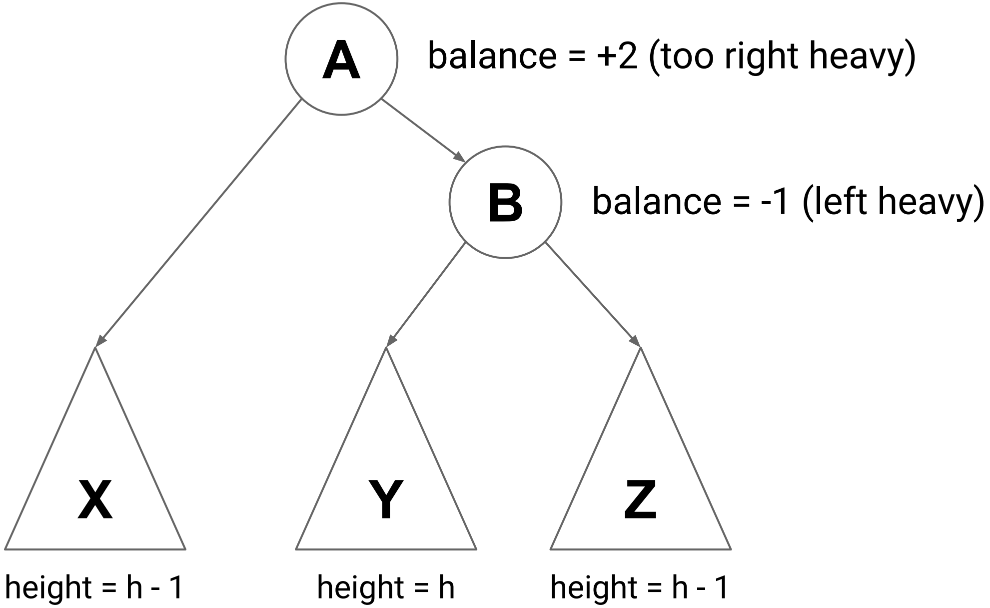
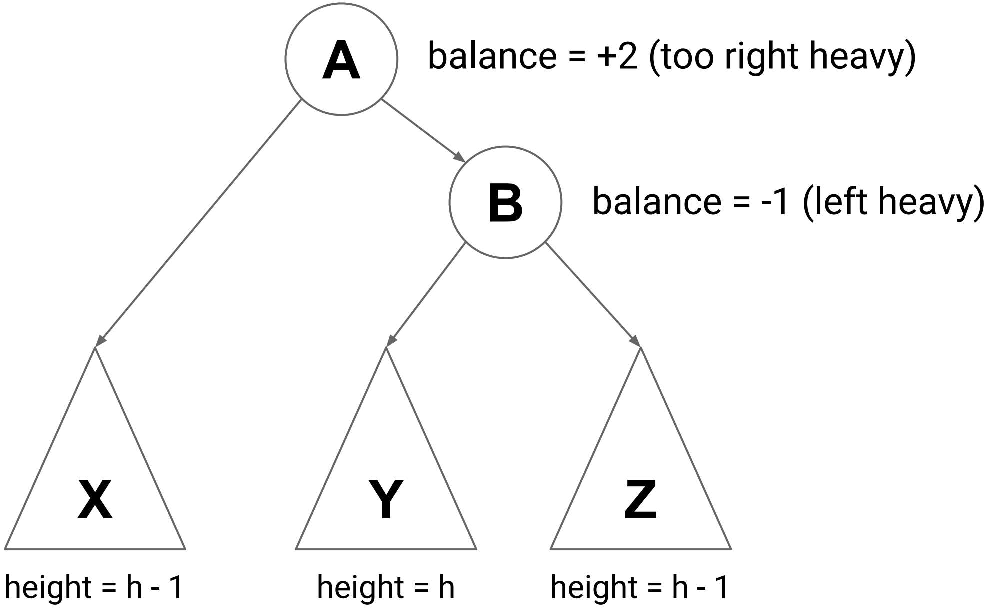
Enforcing the AVL Constraint: Case 3



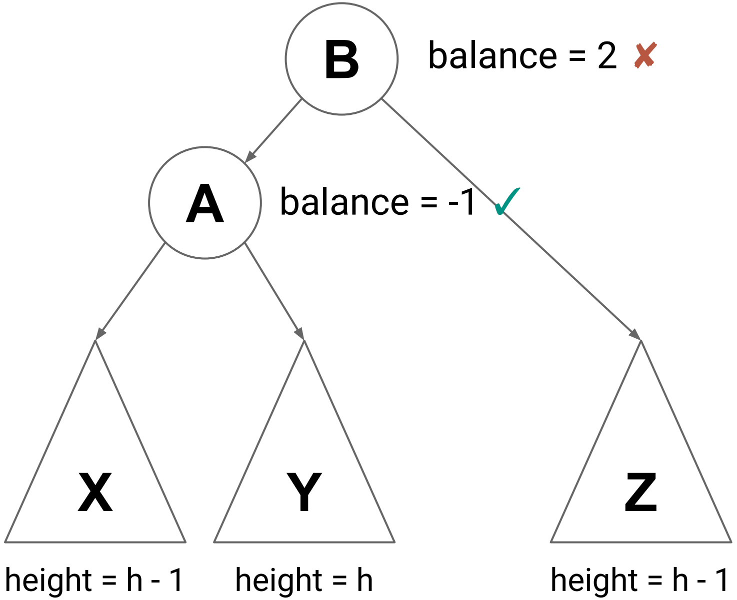
How can we fix this?

Enforcing the AVL Constraint: Case 3



How can we fix this?
Will just a single left rotation work?

Enforcing the AVL Constraint: Case 3



How can we fix this?
Will just a single left rotation work? **No**

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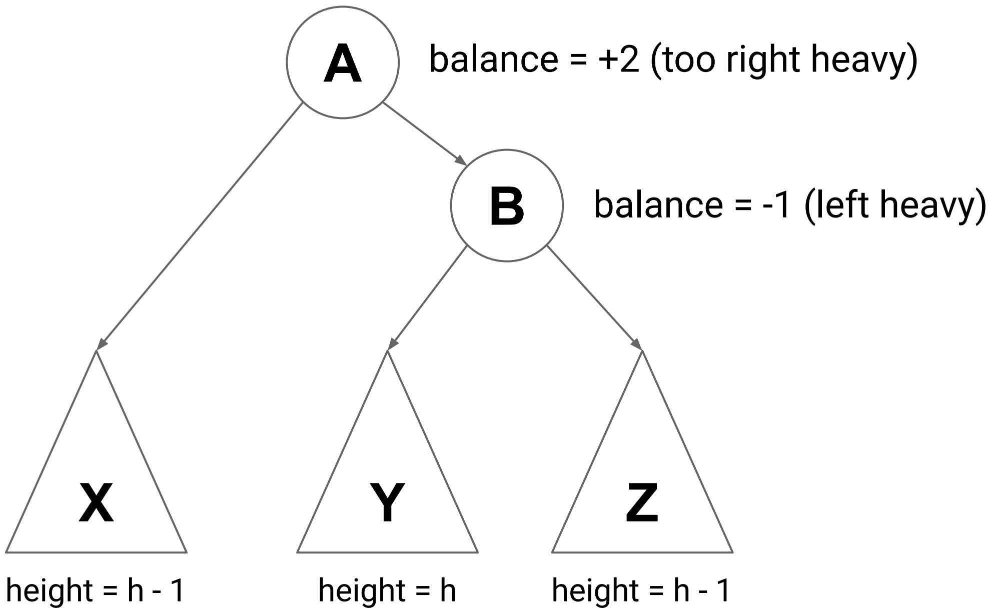

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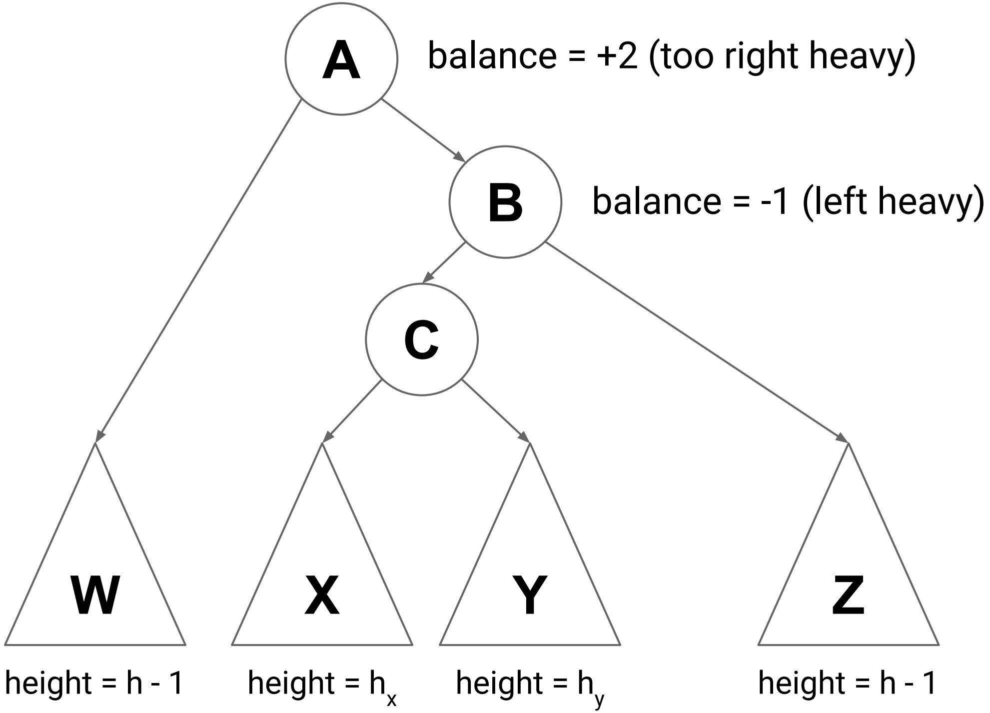

How can we fix this?

Height of **C** we know must be ***h***

Therefore At least one of $\boldsymbol{h_x}$ or $\boldsymbol{h_y}$ must be ***h* - 1**

The other can also be ***h* - 2**, or ***h* - 1**

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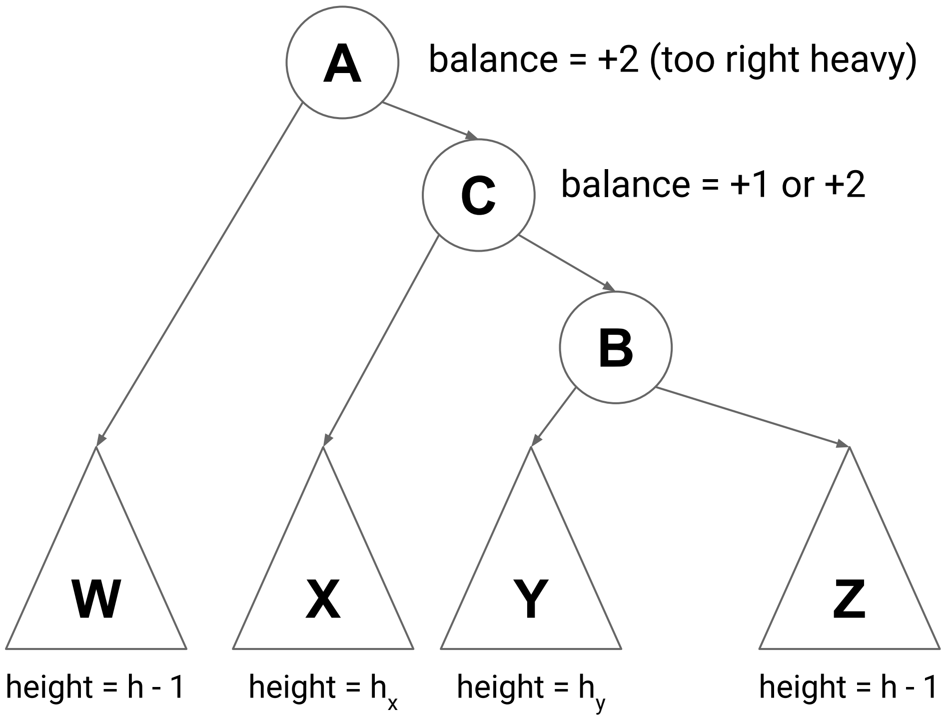

How can we fix this?
Rotate right first: `rotate(B,C)`

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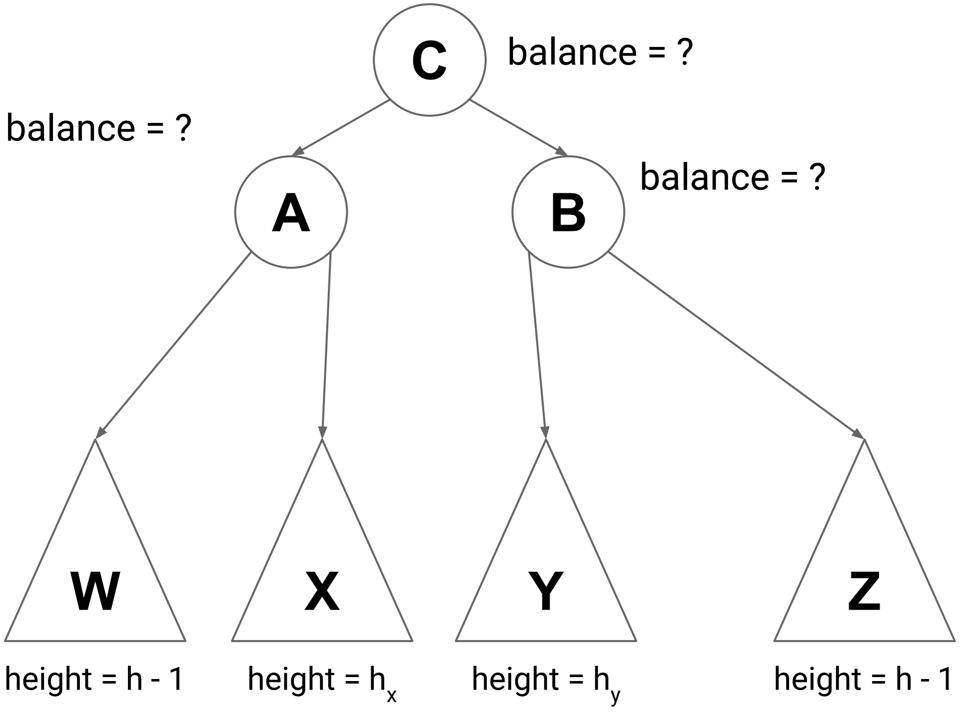

How can we fix this?
Rotate right first: **`rotate(B,C)`**
Then right left: **`rotate(A,C)`**

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Therefore At least one of $\boldsymbol{h_x}$ or $\boldsymbol{h_y}$ must be ***h* - 1**

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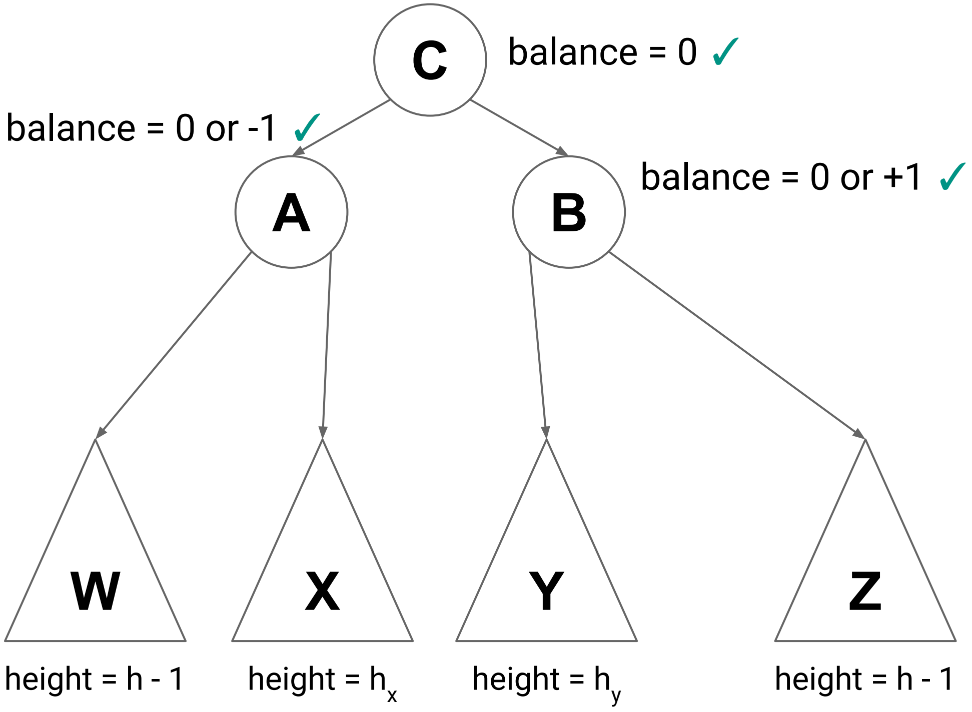

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Rotate right first: **`rotate(B,C)`**
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Enforcing the AVL Constraint

- If too right heavy (balance == +2)
  - If right child is right heavy (balance == +1) or balanced (balance == 0)
    - rotate left around the root
  - If right child is left heavy (balance == -1)
    - rotate right around root of right child, then rotate left around root
- If too left heavy (balance == -2)
  - Same as above but flipped

Therefore if we have a balance factor that is off, but all children are AVL trees, we can fix the balance factor in at most 2 rotations

Inserting Records

To insert a record into an AVL Tree:

1. Find the insertion point (remember it is a BST)
2. Insert the new leaf and set balance factor to 0
3. Trace path back up to root and update balance factors
    a. If a balance factor becomes +/-2 then rotate to fix

Inserting Records

To insert a record into an AVL Tree:

1. Find the insertion point (remember it is a BST) — $\boldsymbol{O(d) = O(\log n)}$
2. Insert the new leaf and set balance factor to 0 — $\boldsymbol{O(1)}$
3. Trace path back up to root and update balance factors — $\boldsymbol{O(d) = O(\log n)}$
    a. If a balance factor becomes +/-2 then rotate to fix — $\boldsymbol{O(1)}$

Inserting Records

```
def insert[K, V](key: K, value: V, root: AVLNode[K, V]): Unit = {
  var node = findInsertionPoint(key, root)
  node._key = key; node._value = value
  node._isLeftHeavy = node._isRightHeavy = false
  while(node._parent.isDefined){
    if(node._parent._left == node){
      if(node._parent._isRightHeavy){
        node._parent._isRightHeavy = false; return
      } else if(node._parent._isLeftHeavy) {
        if(node._isLeftHeavy){ node._parent.rotateRight() }
        else { node._parent.rotateLeftRight() }
        return
      } else {
        node._parent.isLeftHeavy = true
      }
    } else { /* symmetric to above */ }
    node = node._parent
  }
}
```

Inserting Records

```scala
def insert[K, V](key: K, value: V, root: AVLNode[K, V]): Unit = {
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Find insertion point and create the new leaf ***O*(*d*) = *O*(log *n*)**

Inserting Records

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Find insertion point and create the new leaf $O(d) = O(\log n)$

$O(d) = O(\log n)$ iterations

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Find insertion point and create the new leaf ***O***(***d***) = ***O***(**log** ***n***)

O(***d***) = ***O***(**log** ***n***) iterations

O(**1**) per iteration

Removing Records

- Removal follows essentially the same process as insertion