CSE 250
Data Structures

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HashTables

Picking a Hash Function

What function could we use that would evenly distribute values to buckets?

Picking a Hash Function

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Wacky Idea: Have $h(x)$ return a random value in **[0,N)**

(This makes apply impossible…but bear with me)

Random Hash Function

$$n = \text{number of elements in any bucket}$$

$$N = \text{number of buckets}$$

$$b_{i,j} = \begin{cases} 1 & \textbf{if} \text{ element } i \text{ is assigned to bucket } j \\ 0 & otherwise \end{cases}$$

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Only true if $b_{i,j}$ and $b_{i',j}$ are uncorrelated for any $i \neq i'$

(h(i) can't be related to h(i'))

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The **expected** number of elements in any bucket j

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$$\mathbb{E}\left[\sum_{i=0}^{n} b_{i,j}\right] = \frac{n}{N}$$

The **expected** number of elements in any bucket j

...given this information, what do the runtimes of our operations look like?

Random Hash Function

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$$b_{i,j} = \begin{cases} 1 & \textbf{if } \text{element } i \text{ is assigned to bucket } j \\ 0 & otherwise \end{cases}$$

Expected runtime of `insert`, `apply`, `remove`: ***O*(*n*/*N*)**

Worst-Case runtime of `insert`, `apply`, `remove`: ***O*(*n*)**

Hash Functions In the Real-World

Examples

- SHA256 ← Used by GIT
- MD5, BCRYPT ← Used by unix login, apt
- MurmurHash3 ← Used by Scala

hash(*x*) is pseudo-random

- **hash(*x*)** ~ uniform random value in [0, INT_MAX)
- **hash(*x*)** always returns the same value for the same ***x***
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We then use modulus to fit this random value into the size of our hash table

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Refresher on Modulus

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quotient divisor remainder

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7	8	9	10	11	12	13

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14	15	16	17	18	19	20

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If my hash table has 7 buckets, and I insert an element with hash code 73, what bucket would it go in?

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If my hash table has 7 buckets, and I insert an element with hash code 73, what bucket would it go in? **73 % 7 = 3**

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  - They act as if they are uniformly random
    - Will evenly distribute elements to buckets
    - hash(x) is uncorrelated with hash(y)
  - They are deterministic (hash(x) will always return the same value)
- We can use these hash functions to determine which bucket an arbitrary element belongs in in $O(1)$ time
- There are expected to be n/N elements in that bucket
  - So runtime for all operations is **expected $O(1) + O(n/N)$ = expected $O(n)$**

Hash Functions + Buckets

Everything is: $O\left(\frac{n}{N}\right)$

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What do we do when this constraint is violated? **Resize!**

Rehashing

When we insert an element that would exceed the load factor we:

1. Resize the underlying array from N_{old} to N_{new}
2. Rehash all of the elements from their old bucket to their new bucket
    a. Element x moves from hash(x) % N_{old} to hash(x) % N_{new}

Rehashing

Let's say we have a hash table of size 6, and hash(***x***) = 65

What bucket does it belong in?

0	**1**	**2**	**3**	**4**	**5**	**6**

Rehashing

Let's say we have a hash table of size 6, and hash(***x***) = 65

What bucket does it belong in? 65 % 6 = 5

0	1	2	3	4	5 X	6

Now we want to resize the array to size 8. Where do we move ***x***?

0	1	2	3	4	5	6	5	6

Rehashing

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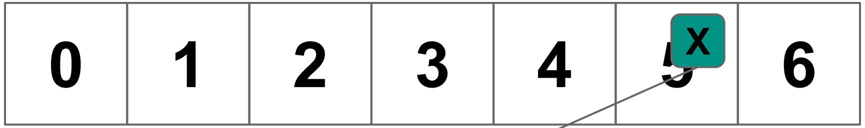
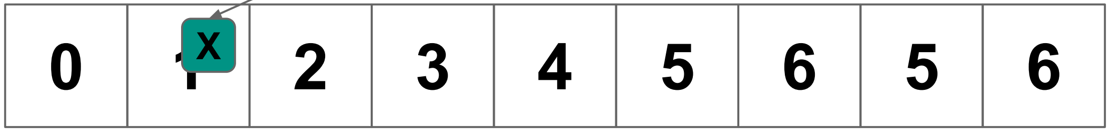


Now we want to resize the array to size 8. Where do we move ***x***? 65 % 8 = 1

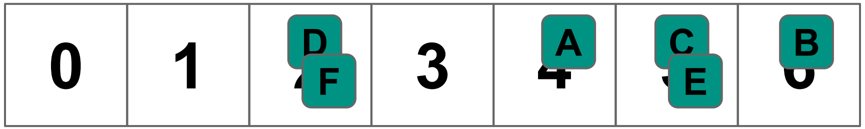


Rehashing

How long will it take to rehash every element after we resize?

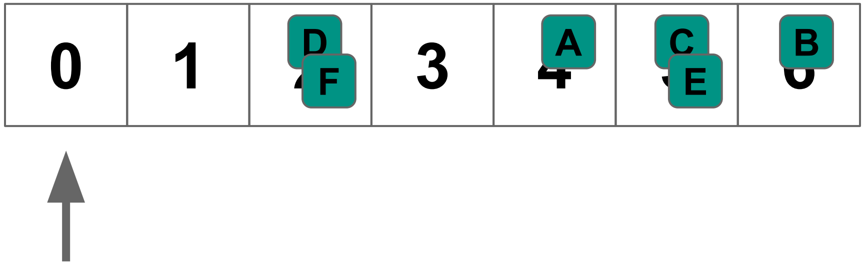
Related Question: *How do we iterate through a hash table?*

Iterating over a Hash Table



Iterating over a Hash Table

Start at the first bucket



Iterating over a Hash Table

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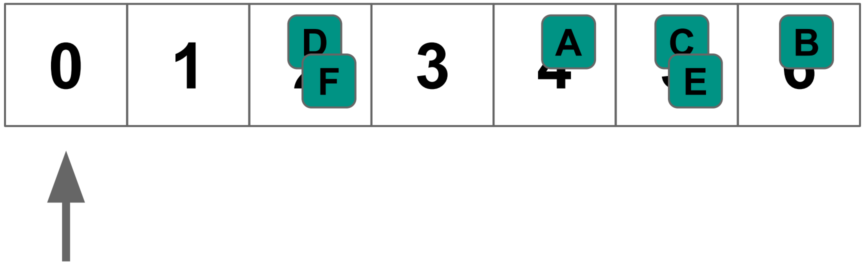
Iterate through that bucket



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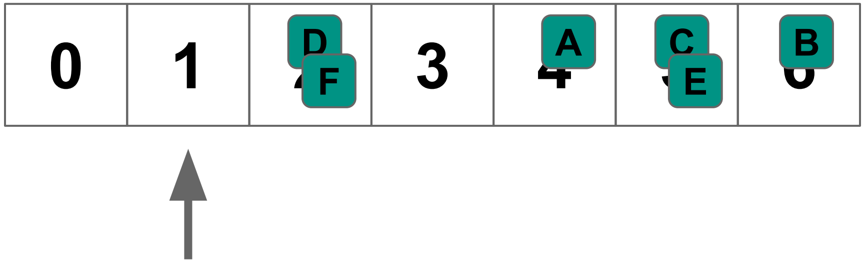

Iterating over a Hash Table

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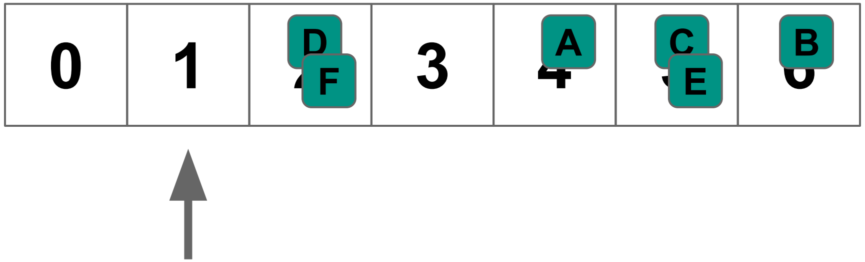
Iterate through that bucket

Move to the next bucket

…and repeat



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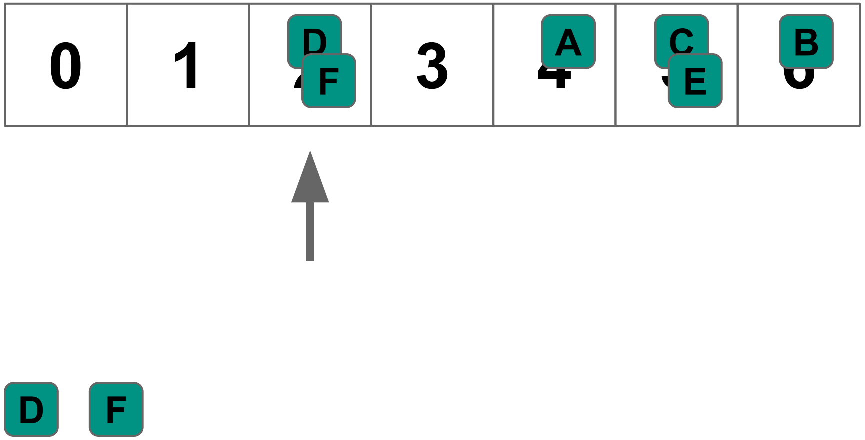

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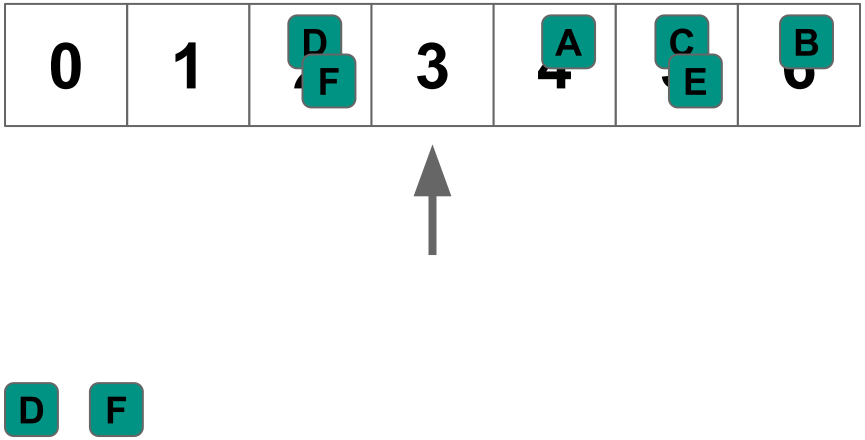

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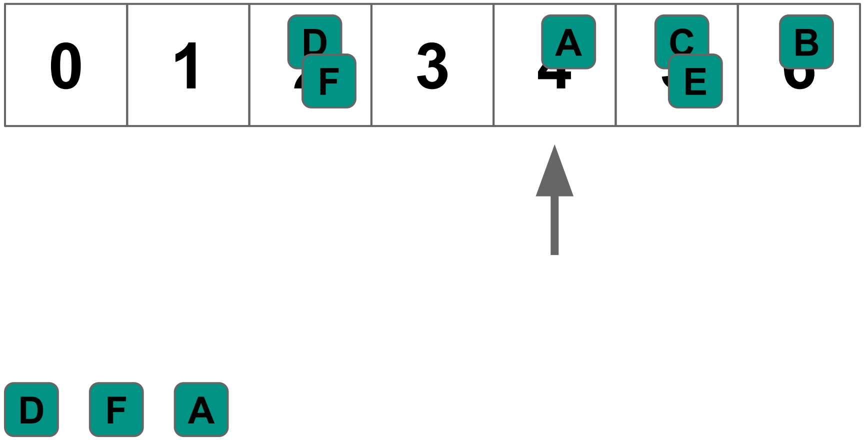

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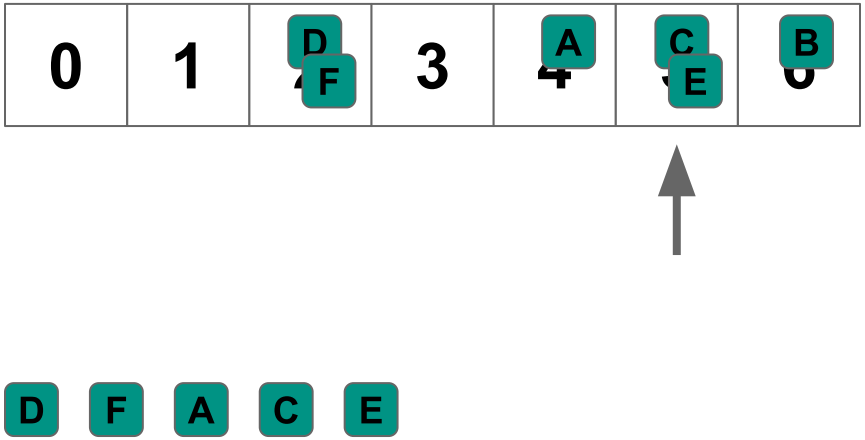

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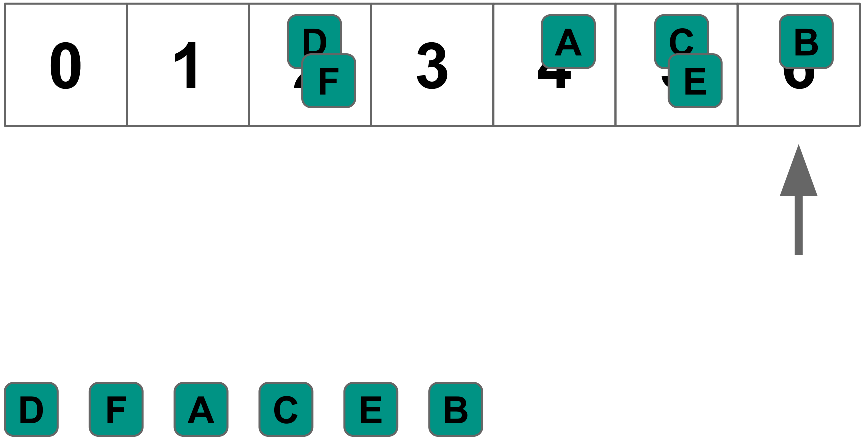

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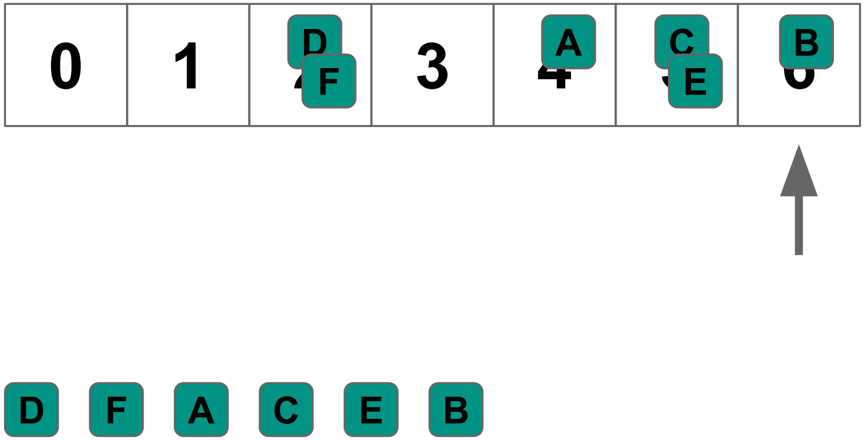

How long does it take?

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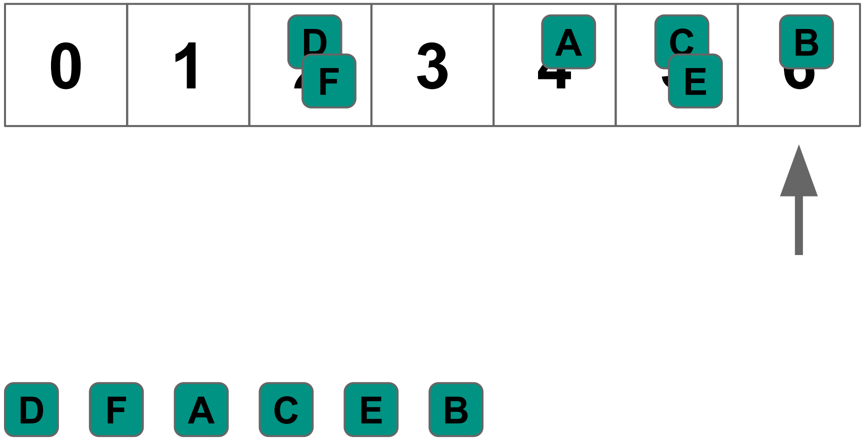

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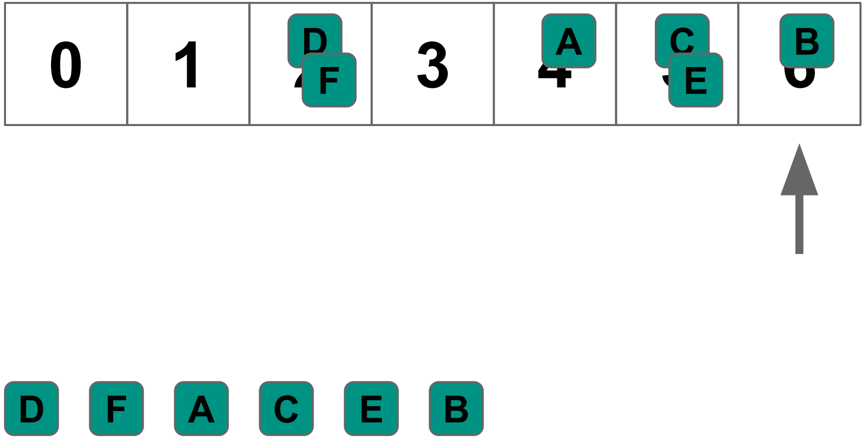

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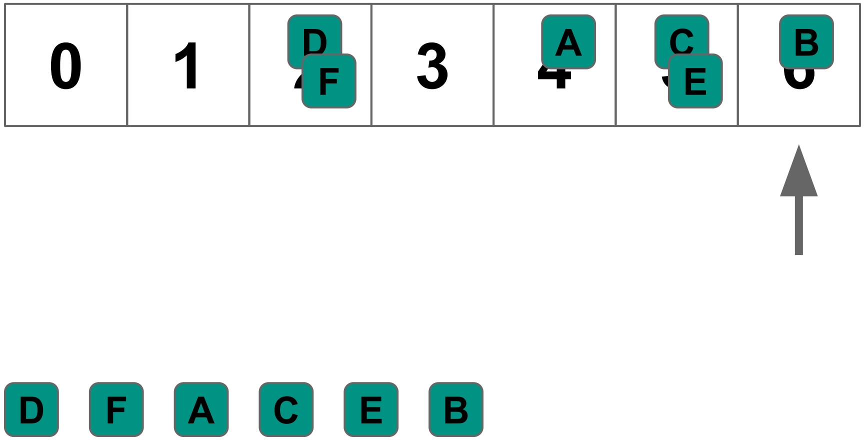

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Visit every element in each bucket

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Iterating through each element costs $\boldsymbol{O(N + n)}$

Rehashing

So how long does it take to rehash an entire hash table with n elements and N buckets?

Rehashing an individual element costs $O(1)$

Iterating through each element costs $O(N + n)$

Rehashing costs: $O(N + n)$

Rehashing

When we insert an element that would exceed the load factor we:

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How long does this take?

1. Allocate the new array: $O(1)$
2. Rehash every element from the old array to the new: $O(N_{old} + n)$
3. Free the old array: $O(1)$

Total: $O(N_{old} + n)$

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Total: $O(N_{old} + n)$

How do we pick N_{new}?

Rehashing

Whenever $\alpha > \alpha_{max}$, double the size of the array (remember ArrayBuffers)

If we start with ***N*** buckets and insert ***n*** elements:

1. First rehash happens at $n_1 = \alpha_{max} \times N$: goes from ***N*** to **2*N***

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If we start with ***N*** buckets and insert ***n*** elements:

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2. Second rehash happens at $n_2 = \alpha_{max} \times 2N$: goes from **2*N*** to **4*N***

Rehashing

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If we start with ***N*** buckets and insert ***n*** elements:

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2. Second rehash happens at $\boldsymbol{n_2} = \alpha_{max} \times \boldsymbol{2N}$: goes from **2*N*** to **4*N***
3. Third rehash happens at $\boldsymbol{n_3} = \alpha_{max} \times \boldsymbol{4N}$: goes from **4*N*** to **8*N***

Rehashing

Whenever $\alpha > \alpha_{max}$, double the size of the array (remember ArrayBuffers)

If we start with $\boldsymbol{N}$ buckets and insert $\boldsymbol{n}$ elements:

1. First rehash happens at $n_1 = \alpha_{max} \times N$: goes from N to $2N$
2. Second rehash happens at $n_2 = \alpha_{max} \times 2N$: goes from $2N$ to $4N$
3. Third rehash happens at $n_3 = \alpha_{max} \times 4N$: goes from $4N$ to $8N$

…

j. jth rehash happens at $n_j = \alpha_{max} \times 2^{j-1}N$: goes from $2^{j-1}N$ to 2^jN

Total Work

With n insertions, choose j s.t. $n = 2^j\alpha_{max}$

$$2^j = n / \alpha_{max}$$

$$j = \log (n / \alpha_{max})$$

$$j = \log(n) - \log(\alpha_{max})$$

$$j \leq \log(n)$$ ← **Number of rehashes**

Total Work

Rehashes required: **≤ log(n)**

The ith rehash: ***O*(2^{i}N)**

$$\sum_{i=0}^{\log(n)} O(2^i N) = O\left(N \sum_{i=0}^{\log(n)} 2^i\right) = O(2^{\log(n)+1} - 1) = O(n)$$

So ***O*(*n*)** work is required to do ***n*** insertions → Insert cost is **amortized *O*(1)**