CSE 250
Data Structures

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The Memory Hierarchy
Announcements

- Autograder for PA4 Tests is now open
LIES!

**Lie #1:** Accessing any element of an array of any length is $O(1)$
- This assumes the "RAM" model of computation
  - Simple, but not perfect
- Real-world hardware isn't this simple
  - Memory is hierarchical
  - Non-Uniform Memory Access (NUMA)

**Lie #2:** The constants don't matter...
Algorithmic Complexity

Remember: $O(f(n))$ placed bounds on growth functions in general. Not necessarily only for runtime growth functions...
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Runtime Bounds (or Runtime Complexity)
- The algorithm takes $O(\ldots)$ time
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- The algorithm needs \( O(...) \) storage
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Runtime Bounds (or Runtime Complexity)
- The algorithm takes $O(...)$ time

Memory Bounds (or Memory Complexity)
- The algorithm needs $O(...)$ storage

I/O Bounds (or I/O Complexity)
- The algorithm performs $O(...)$ accesses to slower memory
The Memory Hierarchy (simplified)

- Cache
- Memory (RAM)
- Solid State Drives (SSDs)
- Hard Disk Drives (HDDs, "Spinning Rust")

Faster → Bigger
The Memory Hierarchy (simplified)

Cache is made of many "Lines" (~64B in size)

Disk is made up of many "Pages" (~4KB in size)
In order to read an Array Entry:

1. Is the array entry in cache?
Reading an Array Entry

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   a. Yes: Return it (1-4 clock cycles)
   b. No: Is it in real memory?
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      i. Yes: Load it into a cache line (10s of cycles)
      ii. No: Load it from a page of virtual memory (100s of cycles)
Reading an Array Entry

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Tiny constant

OK constant

HUGE constant

In practice, these constants do matter!
Ground Rules: Disk vs RAM

1. All data starts off in a file on disk
   a. Need to load data into RAM before accessing it
   b. Load data in 4KB pages
   c. Amount of RAM is finite

2. Must describe 3 features of an algorithm
   a. Number of instructions (runtime complexity)
   b. Number of data loads (I/O complexity)
   c. Number of pages of RAM required (memory complexity)

Note: Similar rules apply to any pair of levels in the hierarchy
Binary Search

Example:

$2^{20}$ records, 64 bytes each (8 byte key, 56 byte value)

64 MB of data total, 16,384 pages, 64 records per page

*How many steps to binary search this data?*
Binary Search

Example:

$2^{20}$ records, 64 bytes each (8 byte key, 56 byte value)

64 MB of data total, 16,384 pages, 64 records per page

How many steps to binary search this data? $\log(2^{20}) = 20$ steps
Binary Search

Example:

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64 MB of data total, 16,384 pages, 64 records per page

Let's assume the target is at position 0

| 16,384 pages |
Binary Search

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Step 0
Load 8192
Binary Search

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64 MB of data total, 16,384 pages, 64 records per page

Let's assume the target is at position 0

16,384 pages

Step 0
Load 8192

Step 1
Load 4096
Binary Search

Example:

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64 MB of data total, 16,384 pages, 64 records per page

Let's assume the target is at position 0

16,384 pages

Step 0
Load 8192

Step 1
Load 4096

Step 2
Load 2048
Example:

$2^{20}$ records, 64 bytes each (8 byte key, 56 byte value)

64 MB of data total, 16,384 pages, 64 records per page

Let's assume the target is at position 0

Page 0 (64 records)

Step 14
($\log(16,384) = 14$)
Load Page 0
Example:

$2^{20}$ records, 64 bytes each (8 byte key, 56 byte value)

64 MB of data total, 16,384 pages, 64 records per page

Let's assume the target is at position 0

Page 0 (64 records)

Step 14

$\log(16,384) = 14$

Load Page 0

Step 15

Already loaded
Binary Search: Complexity

**Steps 0 - 14:** Sloooooow...each one loaded a new page (15 pages loaded)

**Steps 15-19:** Fast! All access the page loaded on step 14

*Runtime complexity* = $O(\log(n))$

*What's the memory complexity?*
Binary Search: Complexity

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What's the memory complexity? \( O(1) \)

How many pages do we need loaded at one time? 1 page... we only care about the maximum memory we will need at any one time
Binary Search: Complexity

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Runtime complexity = $O(\log(n))$

What's the memory complexity? $O(1)$

How many pages do we need loaded at one time? 1 page...we only care about the maximum memory we will need at any one time

What about I/O complexity?
Let's set up some variables:

- \( n \) - total number of records
- \( R \) - record size (in Bytes)
- \( P \) - page size (in Bytes)
- \( C = \lfloor R/P \rfloor \) records per page
Binary Search does log(n) steps broken into two stages:

**Stage 1:** Each request has to load a new page into memory

**Stage 2:** The remaining requests all happen in the same page
Binary Search: I/O Complexity

Binary Search does $\log(n)$ steps broken into two stages:

Stage 1: Each request has to load a new page into memory

Stage 2: The remaining requests all happen in the same page

Remember: Our page size is fixed...$C$ records per page

Therefore: The last $\log(C)$ binary search steps are all on the same page
Binary Search: I/O Complexity

Binary Search does $\log(n)$ steps broken into two stages:

Stage 1: Each request has to load a new page into memory
- $\log(n) - \log(C)$ steps

Stage 2: The remaining requests all happen in the same page
- $\log(C)$ steps

Remember: Our page size is fixed... $C$ records per page
Therefore: The last $\log(C)$ binary search steps are all on the same page
Binary Search: I/O Complexity

Binary Search does $O(\log(n) - \log(C))$ loads from memory

**Therefore:** I/O complexity of Binary Search is $\log(n)$
Binary Search: Complexity

Binary Search Complexity:
- Runtime Complexity: $O(\log(n))$
- Memory Complexity: $O(1)$
- I/O Complexity: $O(\log(n))$

How can we improve on this?
Observations

Observation 1:
- Total size of records: $64MB = 2^{20} \times \text{sizeof(key + data)}$
- Total size of keys only: $8MB = 2^{20} \times \text{sizeof(key)}$

Observation 2:
- The first stage doesn't care what array index the record is at, just the page it is on
- Each page stores a contiguous range of keys...
Fence Pointers

**Idea:** Precompute the greatest key stored on each page
- \( n \) total records, \( C \) records per page, \( n/C \) keys required
- For our example, \( 2^{20} \) records needs \( 2^{14} \) pages, therefore \( 2^{14} \) keys
  - \( 2^{20} \) 64 byte records need 64MB memory
  - \( 2^{14} \) 8 byte keys only needs 512KB memory
- Call this a "Fence Pointer Table" and store it in memory

**RAM:** \( 2^{14} = 16,384 \) keys (Fence Pointer Table)

**Disk:** 16,384 pages (Actual Data)
### Fence Pointer Example

#### Binary Search for 321

<table>
<thead>
<tr>
<th>RAM (Fence Pointer Table):</th>
<th>178</th>
<th>273</th>
<th>412</th>
<th>611</th>
<th>913</th>
<th>975</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

#### Disk:

<table>
<thead>
<tr>
<th>Page 0</th>
<th>Page 1</th>
<th>Page 2</th>
<th>Page 3</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>keys 0 - 178</td>
<td>keys 192 - 273</td>
<td>keys 274 - 412</td>
<td>keys 412 - 611</td>
<td>...</td>
</tr>
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Fence Pointer Example

Binary Search for 321

273 < 312 ≤ 412

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Disk:

- keys 0 - 178
- keys 192 - 273
- keys 274 - 412
- keys 412 - 611

Page 0  Page 1  Page 2  Page 3  ...
Fence Pointer Example

Binary Search for 321

273 < 312 ≤ 412

Load page 2 then binary search it

Disk:
- Page 0: keys 0 - 178
- Page 1: keys 192 - 273
- Page 2: keys 274 - 412
- Page 3: keys 412 - 611

RAM (Fence Pointer Table):
- 178
- 273
- 412
- 611
- 913
- 975
- ...
Binary Search with Fence Pointers

Step 1: Binary search the fence pointer table
  ● $O(\log(n) - \log(C))$ steps
  ● All in memory, 0 disk reads

Step 2: Load page
  ● 1 step, 1 disk read

Step 3: Binary search within page
  ● $O(\log(C))$ steps
  ● All in memory, 0 disk reads
Binary Search with Fence Pointers

**Step 1:** Binary search the fence pointer table
- $O(\log(n) - \log(C))$ steps
- All in memory, 0 disk reads

**Runtime:** $O(\log(n))$

**Step 2:** Load page
- 1 step, 1 disk read

**I/O:** $O(1)$

**Step 3:** Binary search within page
- $O(\log(C))$ steps
- All in memory, 0 disk reads

**Memory?**
Binomial Search with Fence Pointers

**Step 1:** Binary search the fence pointer table
- $O(\log(n) - \log(C))$ steps
- All in memory, 0 disk reads

**Step 2:** Load page
- 1 step, 1 disk read

**Step 3:** Binary search within page
- $O(\log(C))$ steps
- All in memory, 0 disk reads

**Runtime:** $O(\log(n))$

**I/O:** $O(1)$

**Memory:** $O(n)$

*We need the entire fence pointer table in memory at all times :(***
What about Runtime/Memory Complexity?

Records per page, $C$, is a constant, size of the fence pointer table is $n / C$.
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Records per page, $C$, is a constant, size of the fence pointer table is $n / C$

Runtime Complexity: $\log(n/C) + \log(C) = O(\log(n))$

- Search the fence pointer table, then search the page
What about Runtime/Memory Complexity?

Records per page, $C$, is a constant, size of the fence pointer table is $\frac{n}{C}$

**Runtime Complexity:** $\log\left(\frac{n}{C}\right) + \log(C) = O(\log(n))$
- Search the fence pointer table, then search the page

**I/O Complexity:** 1 page read = $O(1)$
- Load the single page found by searching the fence pointer table
What about Runtime/Memory Complexity?

Records per page, \( C \), is a constant, size of the fence pointer table is \( n / C \)

Runtime Complexity: \( \log(n/C) + \log(C) = O(\log(n)) \)
- Search the fence pointer table, then search the page

I/O Complexity: 1 page read = \( O(1) \)
- Load the single page found by searching the fence pointer table

Memory Complexity: \( O(n/C + C) = O(n) \)
- Need to store the fence pointer table (at all times), and one additional page that we load after the fence pointer table search
What about Runtime/Memory Complexity?

Records per page, $C$, is a constant, size of the fence pointer table is $n / C$

Runtime Complexity: $\log(n/C) + \log(C) = O(\log(n))$
- Search the fence pointer table, then search the page

I/O Complexity: $O(n)$ is not ideal... and what if the fence pointer table gets too big for memory?
- Load the single page found by searching the fence pointer table

Memory Complexity: $O(n/C + C) = O(n)$
- Need to store the fence pointer table (at all times), and one additional page that we load after the fence pointer table search
Improving on Fence Pointers

At some point, we will have to store the fence pointers on Disk...

In our current example with 4KB pages, and 8B keys, we can fit 512 keys per page
At some point, we will have to store the fence pointers on Disk...

In our current example with 4KB pages, and 8B keys, we can fit **512 keys per page**

**Idea:** What if we binary search the fence pointers on disk?
Improving on Fence Pointers

With our current example:

- We can store 512 8B keys per 4KB page ($2^9$ keys per page)
- $2^{20}$ records / 64 records per page = $2^{14}$ pages of records
- $2^{14}$ fence pointer keys = $2^5$ pages
- Binary search of the pointer key pages will require $\log(2^5) = 5$ loads

In general: $\log(n) - \log(C) - \log(\text{keys/page})$
Improving on Fence Pointers

With our current example:

- We can store 512 8B keys per 4KB page (2^9 keys per page)
- \(2^{20}\) records / 64 records per page = \(2^{14}\) pages of records
- \(2^{14}\) fence pointer keys = \(2^5\) pages
- Binary search of the pointer key pages will require \(\log(2^5) = 5\) loads

In general: \(\log(n) - \log(C) - \log(\text{keys/page}) \leftrightarrow \text{Still } O(\log(n))\)
Improving on Fence Pointers

IO Complexity: $\log(n) - \log(C_{data}) - \log(C_{key}) = O(\log(n))$

- $C_{data} = \text{records per page}$ (ie: 64)
- $C_{key} = \text{keys per page}$ (ie: 512)

Can we improve our search of the on-disk Fence Pointer Table...?
Improving on Fence Pointers

**Idea:** A fence pointer table for our fence pointer table!
(and if that fence pointer table is too big...a fence pointer table for that table...and so on and so on and so on and so on...until we have one that fits in memory)
Improving on Fence Pointers

- Fence pointer array (in memory)
- Page of actual data
Improving on Fence Pointers

1. Binary Search FP Table to find page

- Fence pointer array (in memory)
- Page of actual data
Improving on Fence Pointers

1. Binary Search FP Table to find page
2. Load page and binary search for record
Improving on Fence Pointers

- Fence pointer array (in memory)
- Fence pointer array (in a page on disk)
- Page of actual data
Improving on Fence Pointers

1. Binary Search @ Level 0 to find Level 1 page
Improving on Fence Pointers

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2. Load and search Level 1 page to find data page
Improving on Fence Pointers

1. Binary Search @ Level 0 to find Level 1 page

2. Load and search Level 1 page to find data page

3. Load and search data page for the record
Improving on Fence Pointers

- Fence pointer array (in memory)
- Fence pointer array (in a page on disk)
- Page of actual data
Improving on Fence Pointers

1. Binary Search @ Level 0 to find Level 1 page
Improving on Fence Pointers

1. Binary Search @ Level 0 to find Level 1 page

2. Load and search Level 1 page to find Level 2 page

- Fence pointer array (in memory)
- Fence pointer array (in a page on disk)
- Page of actual data
Improving on Fence Pointers

1. Binary Search @ Level 0 to find Level 1 page

2. Load and search Level 1 page to find Level 2 page

3. Load and search Level 2 page to find data page

Fence pointer array (in memory)

Fence pointer array (in a page on disk)

Page of actual data
Improving on Fence Pointers

1. Binary Search @ Level 0 to find Level 1 page
2. Load and search Level 1 page to find Level 2 page
3. Load and search Level 2 page to find data page
4. Load and search data page to find the record

Fence pointer array (in memory)
Fence pointer array (in a page on disk)
Page of actual data
Improving on Fence Pointers ISAM Index

1. Binary Search @ Level 0 to find Level 1 page

2. Load and search Level 1 page to find Level 2 page

3. Load and search Level 2 page to find data page

4. Load and search data page to find the record
ISAM Index

IO Complexity:
- 1 read at L0 (or assume already in memory)
- 1 read at L1
- 1 read at L2
- ...
- 1 read at $L_{max}$
- 1 read at data level
How many levels will there be (this isn't a binary tree...)}
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- Level 0: 1 page with $C_{key}$ keys
ISAM Index

How many levels will there be (this isn't a binary tree...)

- Level 0: 1 page w/ $C_{key}$ keys
- Level 1: Up to $C_{key}$ pages w/ $C_{key}^2$ keys
ISAM Index

How many levels will there be (this isn't a binary tree...)

- Level 0: 1 page w/ $C_{key}$ keys
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- ...
ISAM Index

How many levels will there be (this isn't a binary tree...)

- Level 0: 1 page with $C_{key}$ keys
- Level 1: Up to $C_{key}$ pages with $C_{key}^2$ keys
- Level 2: Up to $C_{key}^2$ pages with $C_{key}^3$ keys
- ...
- Level max: Up to $C_{key}^{max}$ pages with $C_{key}^{max+1}$ keys
How many levels will there be (this isn't a binary tree...)

- Level 0: 1 page w/ $C_{key}$ keys
- Level 1: Up to $C_{key}^2$ pages w/ $C_{key}^2$ keys
- Level 2: Up to $C_{key}^2$ pages w/ $C_{key}^3$ keys
- ... 
- Level max: Up to $C_{key}^{max}$ pages w/ $C_{key}^{max+1}$ keys
- Data Level: Up to $C_{key}^{max+1}$ pages w/ $C_{data} C_{key}^{max+1}$ records
ISAM Index

\[ n = C_{\text{data}} C_{\text{key}}^{\text{max} + 1} \]
ISAM Index

\[ n = \frac{C_{\text{data}}}{C_{\text{data}}} \cdot C_{\text{key}}^{\text{max} + 1} \]

\[ \frac{n}{C_{\text{data}}} = C_{\text{key}}^{\text{max} + 1} \]
ISAM Index

\[ n = C_{data}C_{key}^{\text{max}+1} \]

\[ \frac{n}{C_{data}} = C_{key}^{\text{max}+1} \]

\[ \log_{C_{key}} \left( \frac{n}{C_{data}} \right) = \text{max} + 1 \]
ISAM Index

\[ n = C_{data} C_{key}^{max + 1} \]

\[ \frac{n}{C_{data}} = C_{key}^{max + 1} \]

\[ \log_{C_{key}} \left( \frac{n}{C_{data}} \right) = max + 1 \]

\[ \log_{C_{key}} (n) - \log_{C_{key}} (C_{data}) = max + 1 \]
ISAM Index

\[ n = C_{data}C_{key}^{\text{max}+1} \]

\[ \frac{n}{C_{data}} = C_{key}^{\text{max}+1} \]

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\[ \log_{C_{key}} (n) - \log_{C_{key}} (C_{data}) = \text{max} + 1 \]

**Number of Levels:** \( O \left( \log_{C_{key}} (n) \right) \)
ISAM Index

\[ n = C_{data} C_{key}^{max+1} \]

\[ \frac{n}{C_{data}} = C_{key}^{max+1} \]

\[ \log_{C_{key}} \left( \frac{n}{C_{data}} \right) = max + 1 \]

\[ \log_{C_{key}} (n) - \log_{C_{key}} (C_{data}) = max + 1 \]

Note this isn't base 2!

Number of Levels: \[ O \left( \log_{C_{key}} (n) \right) \]
ISAM Index

Like Binary Search, but "Cache-Friendly"
● Still takes $O(\log(n))$ steps
● Still requires $O(1)$ memory (1 page at a time)
● Now requires $\log_{c_{\text{key}}}(n)$ loads from disk ($\log_{c_{\text{key}}}(n) \ll \log_2(n)$)
What if the data changes?