Mar 9

Dijkstra's algo

$$d'(w) = \min_{\substack{u \in R \\ u,w \in E}} \{d(u) + \ell_{u,w}$$

0. $R = \{s\}$, $d(s) = 0$

1. While $\exists\, x \notin R$ s.t. $\exists\, u \in R$ with $(u,x) \in E$

   Pick w among all such x's with smallest $d'(w)$ value

   Add w to R

   $d(w) = d'(w)$

Def: Let P_u be the s-u path in "Dijkstra tree"

THM: $\forall\, u \in V$, P_u is the shortest s-u path

$\Longrightarrow d(u)$ are computed correctly $\Longrightarrow$ Dijkstra is correct

(Ex.)

Lemma 1: At the end of each iteration of the while loop, $\forall\, u \in R$, P_u is ~~the~~ a shortest s-u path.

Lemma 2: $\forall\, u \in V$ s.t. $\exists$ s-u path $\Longleftrightarrow u \in R$ at the end

(Ex.)

Lemmas $1 + 2 \Rightarrow$ THM

Pf idea: By induction on $|R|$

Lemma 1

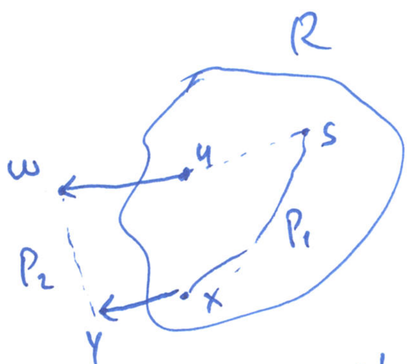
Base case: $|R|=1$, $R=\{s\}$, $d(s)=0$

I.H.: Assume Lemma is true $|R|=k$ $(k\geq 1)$

I.S. Argue $|R|=k+1$

Assume w is $(k+1)^{th}$ vertex added to R

Assume w is "discovered" from u $\left(d(w)=d(u)+\ell_{(u,w)} \right)$



$P_w = P_u, w$

Goal: ~~Pw~~ Argue P_w is a shortest-path s-w

Pf: By contradiction.

$\exists$ a s-w path P'_w s.t. $\ell(P'_w) < \ell(P_w)$ $(*)$

As $s \in R$ and $w \notin R \Rightarrow P'_w$ "crosses" R at some point

$\Rightarrow \exists\, x \in R,\ y \notin R$ s.t. $(x,y) \in E$

$$P'_w = P_1, x, y, P_2 \Rightarrow \ell(P'_w) = \ell(P_1) + \ell_{(x,y)} + \ell(P_2)$$

$$\text{def of } d(x) \rightarrow \geq \underbrace{d(x) + \ell_{(x,y)}} + \ell(P_2)$$

$$\text{def of } d'() \rightarrow \geq d'(y) + \ell(P_2)$$

$$\geq d'(y) \underset{\substack{\text{since } w \\ \text{is chosen before } y}}{\geq} d'(w) \underset{\text{algo defn}}{=} d(w) = \ell(P_w)$$

$\Rightarrow \ell(P'_w) \geq \ell(P_w) \Rightarrow$ contradiction with $(*)$ ∎