Mar 23

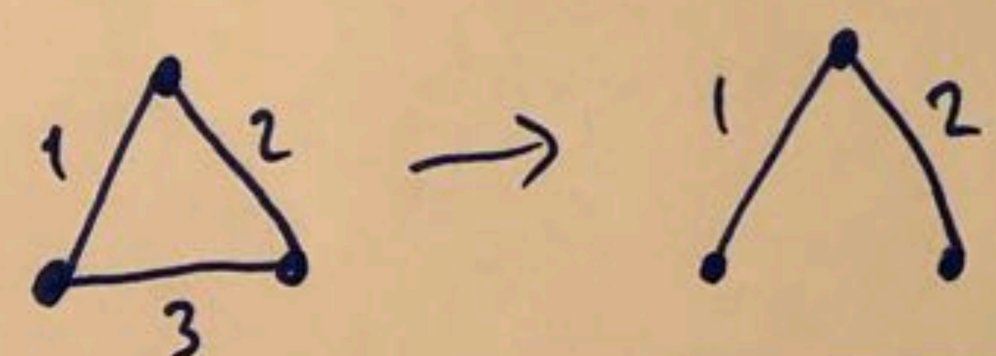
Minimum Spanning Tree (MST)

Input: $G = (V, E)$, $c_e \geqslant 0 \quad \forall e \in E$

connected, undirected [for convenience only]

Output: (i) $E' \subseteq E$ s.t. $T = (V, E')$ is connected (sub-graph of G)

(ii) $\min c(T) = \sum_{e \in E'} c_e$



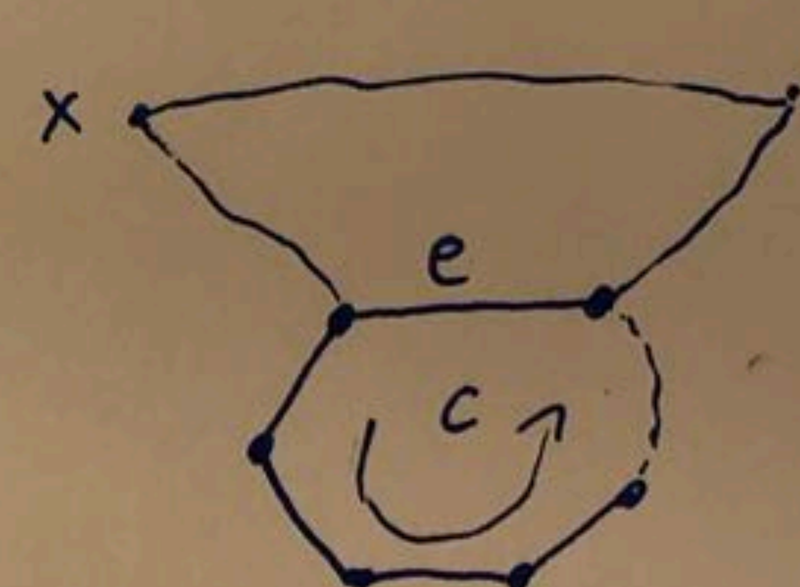
PROP Let $c_e > 0 \quad \forall e \in E$. Then the optimal solution T is a tree.

(Pf idea) By contradiction

Assume T is <u>NOT</u> a tree

$\Longrightarrow$ $\exists$ a cycle in T.

as T is connected & undirected



Fix any cycle c in T

Fix any edge e in c

Delete e from T to get $T' = (V, E' \setminus \{e\})$

Claim 1: T' is connected ; **Claim 2:** $c(T') < c(T)$

$\Rightarrow$ Claim 1+2 $\Rightarrow$ contradiction (optimality of T')

Pf of Claim 2: $c(T') = c(T) - c_e < c(T)$ as $c_e > 0$

Pf of Claim 1: Consider $x, y \in V$

Case 1: $\exists$ an x-y path that does not use $e \Rightarrow x$ & y are connected in T'

Case 2: All x-y path use edge $e \Rightarrow$ use the rest of c instead of e

$\Downarrow$

(x, y) connected in T'

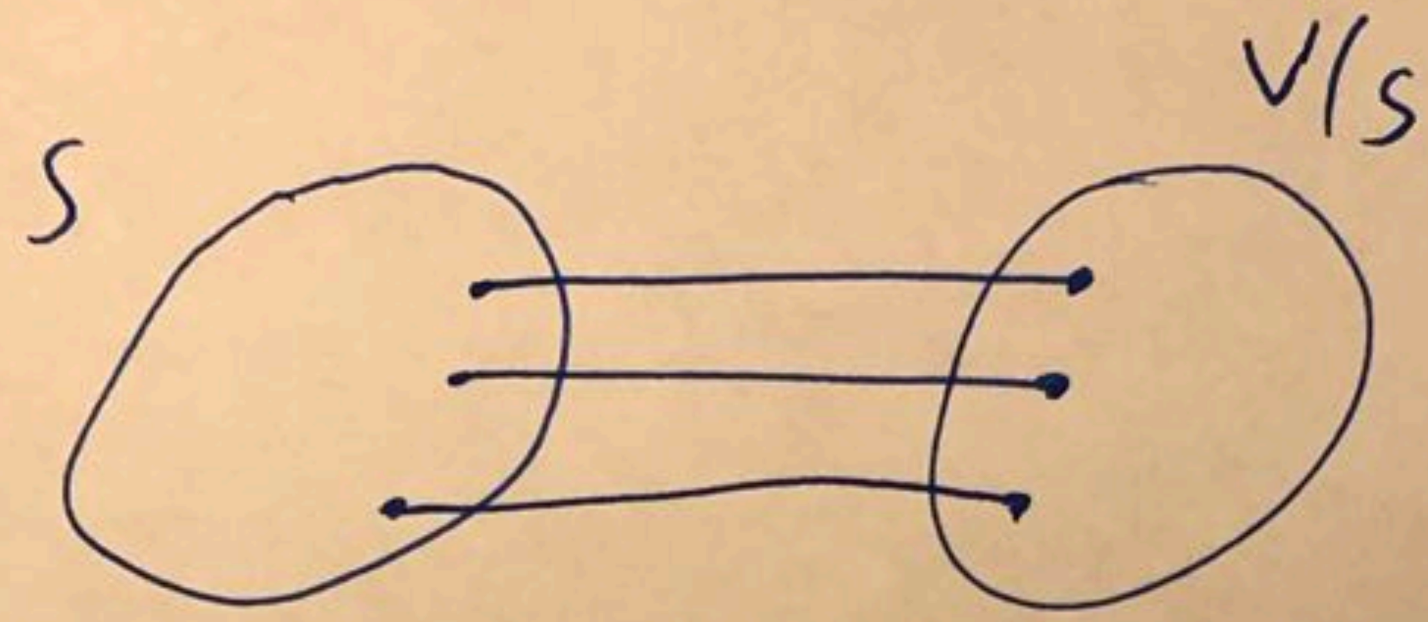
Cut Property Lemma

$S \neq V$

<u>Assume</u>: All c_e's are distinct

For all cuts $(S, V/S)$ s.t. $S \neq \emptyset$, $V/S \neq \emptyset$

Consider all cutting edges



Let $\underline{\underline{e}}$ be the cheapest crossing edge

$\Longrightarrow$ e is in <u>ALL</u> MSTs