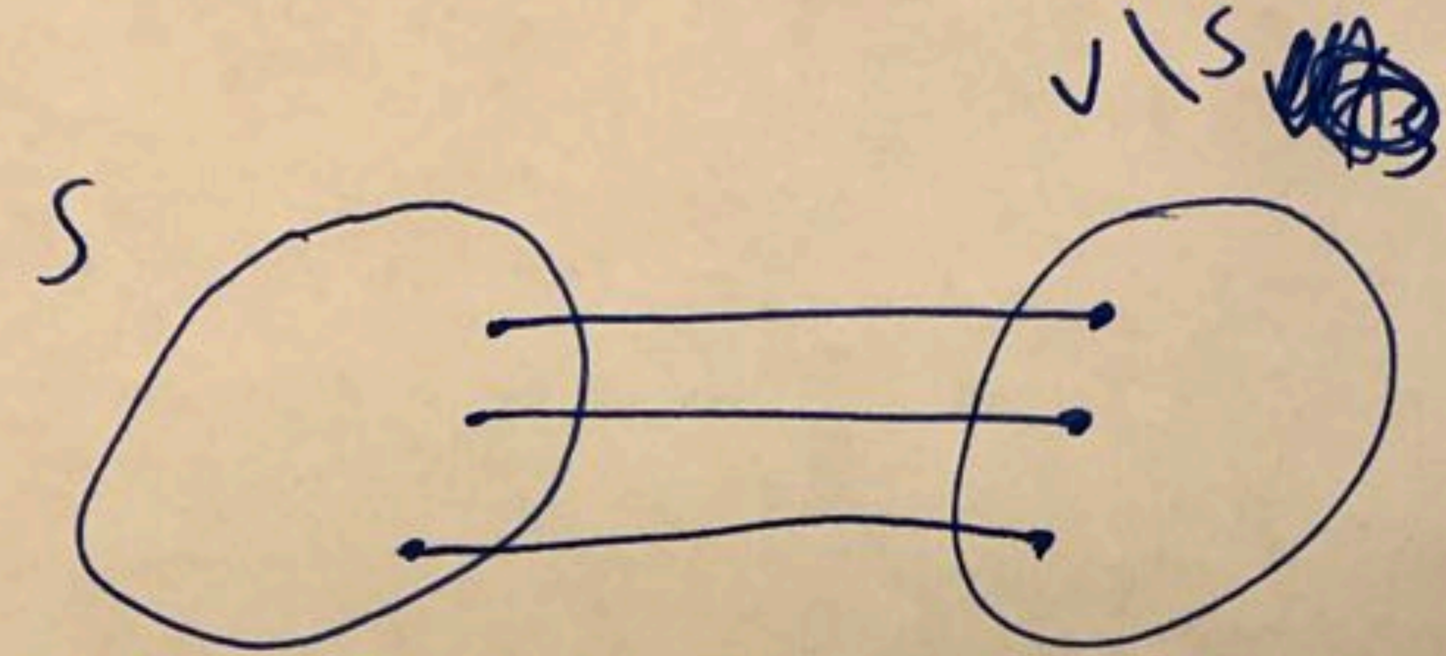
Cut Property Lemma

<u>Assume</u>: All c_e's are distinct

$S \neq V$

$\Updownarrow$

Forall cuts $(S, V \setminus S)$ s.t. $S \neq \emptyset$, $V \setminus S \neq \emptyset$



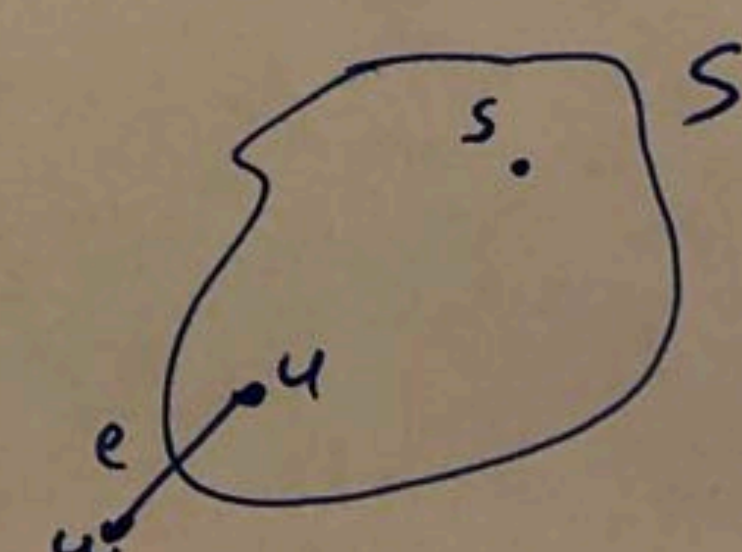
Consider all cutting edges

Let $\underline{\underline{e}}$ be the cheapest crossing edge

$\Longrightarrow$ e is in <u>ALL</u> MSTs

Mar 25 Assume cut-property lemma is true (c_e's are distinct)

<u>THM 1</u>: Prim algo is correct



<u>Pf(idea)</u>: Consider the run of the algo. when it is about to add e to T.

<u>Goal</u>: e is the cheapest crossing edge for <u>some</u> cut $(S, V \setminus S)$ $\xRightarrow{\text{cut property lemma}}$ this is a "safe" choice.

Apply the cut property lemma on $(S, V \setminus S)$ where S is from Prim's alg.

<u>Claim 1</u>: $S \neq \emptyset$ $(u \in S)$

<u>Claim 2</u>: $S \neq V$ $(w \notin S)$

<u>Claim 3</u>: e is the cheapest crossing edge (follows from algo statement)

$\Longrightarrow$ every edge added by Prim is correct/safe.

Claim 4: At the end of each iteration (S, T) is connected.

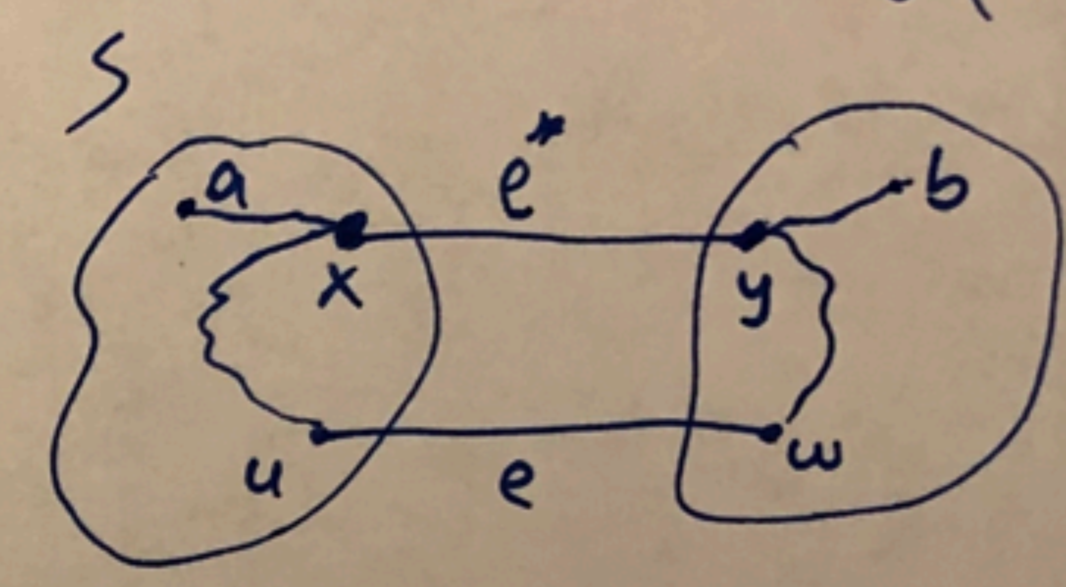
$\Rightarrow$ at the end of the algo (V, T) is connected.

$\boxed{\text{Pf: Ex}}$

Claims 1+2+3+4 $\Rightarrow$ Thm 1 ∎

Proof of Cut-Property Lemma

(Pf idea): By contradiction



Assume not $\Rightarrow \exists$ a cut $(S, V \setminus S)$ s.t. e is the cheapest crossing edge,

$\exists$ an MST T s.t. $e \notin T$.

Since T is connected $\Rightarrow \exists$ u,w path in T

$u \in S, w \notin S \Rightarrow \exists\ x \in S, y \notin S$ s.t. $(x,y) = e^*$ is an edge in T.

Define: $T' = (T \setminus \{e^*\}) \cup \{e\}$

Claim 1: $c(T') = c(T) - C_{e^*} + C_e < c(T)$ ← as $C_{e^*} > C_e$

Claim 2: T' is connected

Case 1: a-b path doesn't use e^* $\Rightarrow$ ✓

Case 2: a-b path does use e^* $\Rightarrow$ take "scenic route" ✓

Claims 1+2 $\Rightarrow$ T is NOT an MST $\Rightarrow$ contradiction ∎

THM: Kruskal's algo is correct.

(consider all edges in increasing order of c_e & add e if adding it does not introduce a cycle)

Pf idea: Consider the case when $e = (u, w)$ is being added to T

Goal: Show e is the cheapest crossing edge for SOME cut $(S, V \setminus S)$

Q: What is S?

A: Let S be set of vertices connected to u using ONLY edges in T so far.

Perturbation Trick

Assume: Assume all c_e's are integers (Ex: without this assumption)

Idea: Add to i^{th} edge an extra $\frac{i}{2.m.n}$ $1 \le i \le m$

$$c'_{e_i} = c_{e_i} + \frac{i}{2.m.n}$$

$\boxed{\text{Ex: All } c'_e \text{ are distinct}}$

Q: By how much does the MST cost change?

edges in tree T is $= n-1$

$\Rightarrow$ max change in $c(T) \le (n-1) \cdot \frac{m}{2m.n}$

$$= \frac{n-1}{2n} < \frac{n}{2n} = \frac{1}{2}$$

$\Rightarrow$ cannot "confuse" 2 spanning trees of diff costs since

if $c(T_1) \neq c(T_2) \Rightarrow |c(T_1) - c(T_2)| \geq 1$

as all c_e are integers.