Mar 30

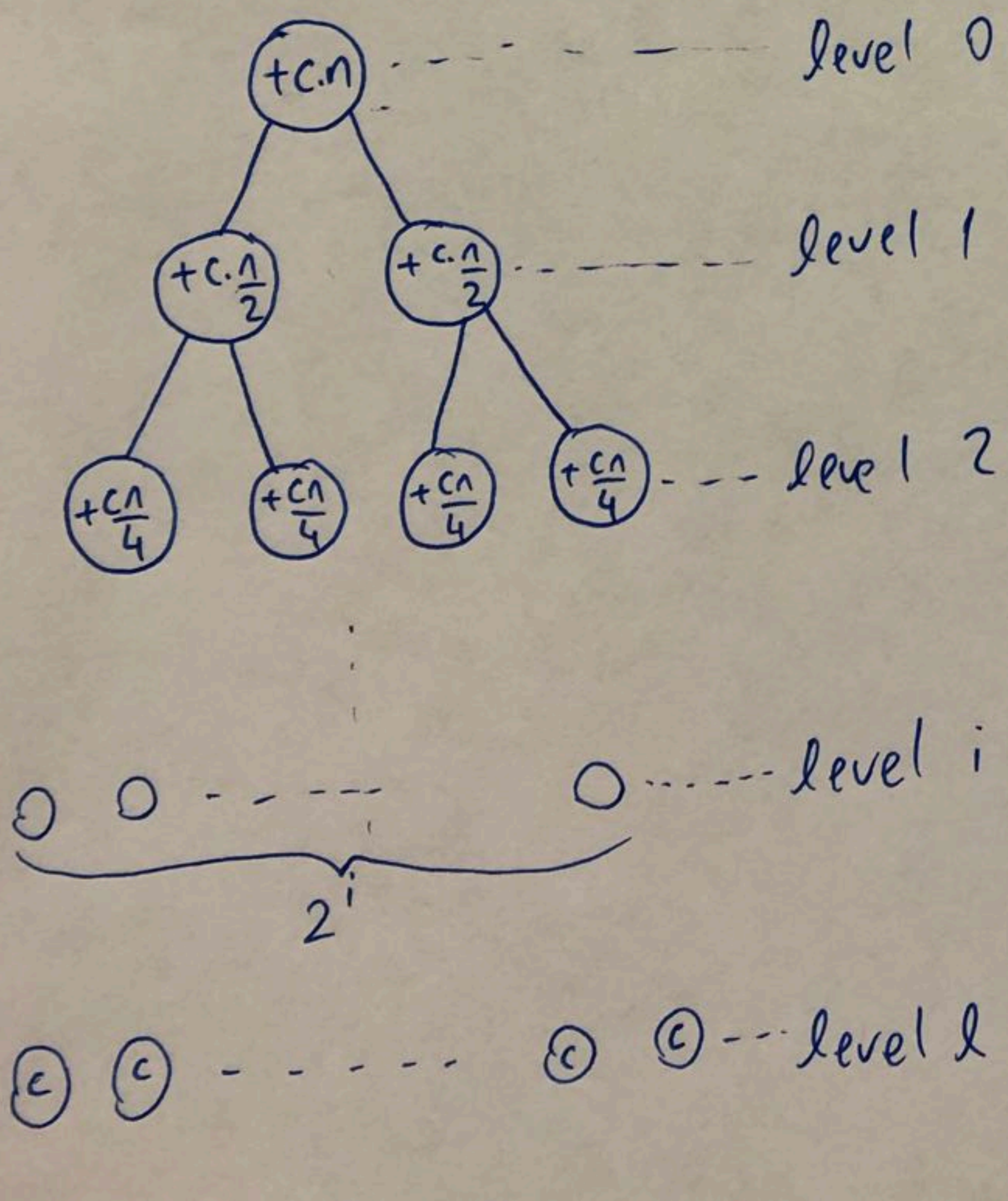
Strategies for solving recurrences

Lemma: $T(n) \leq c.n.\log_2 n + cn$

1. "Unroll" the recurrence and use the pattern
2. Guess the answer and verify using induction on n

Pf of Lemma:

$$T(n) \leq c.n + 2\,T\left(\frac{n}{2}\right)$$
$$\leq c.n + 2\left(\frac{cn}{2} + 2.T\left(\frac{n}{4}\right)\right)$$
$$= c.n + c.n + 4.T\left(\frac{n}{4}\right)$$
$$\leq c.n + c.n + 4.\left(\frac{cn}{4} + 2.T\left(\frac{n}{8}\right)\right)$$
$$= c.n + c.n + c.n + 8.T\left(\frac{n}{8}\right)$$



$\to$ Contribution from level i

$$= 2^i \cdot \frac{cn}{2^i} = c.n$$

$\Rightarrow T(n) \leq c.n.(\#\text{levels})$

Note: $n = 2^l$

$\Rightarrow l = \log_2 n$

$\Rightarrow T(n) \leq c.n.(\log_2 n + 1)$

$= cn.\log_2 n + cn$ ∎

Strategy 2: Guess; $T(n) \leq c.n.\log_2 n + c.n$

Base case: $\Rightarrow T(1) = c.1.\log_2 1 + c$

$= c$ ✓

Inductive hypothesis: $T\left(\frac{n}{2}\right) \leq c.\frac{n}{2}.\log_2\left(\frac{n}{2}\right) + c.\frac{n}{2}$

$= \frac{cn}{2}\left(\log_2\left(\frac{n}{2}\right) + 1\right)$

$= \frac{cn}{2}\left(\log_2 n - \log_2 2 + 1\right)$

$= \frac{cn}{2}.\log_2 n \quad (*)$

Inductive step:

By recursion

$T(n) \leq c.n + 2.T\left(\frac{n}{2}\right)$

by $(*)$ $\leq c.n + 2.\left(\frac{cn}{2}.\log_2 n\right)$

$= cn + cn.\log_2 n$ ∎