Apr 20

Subset sum: n integers $w_1, w_2, \dots w_n$

Budget $W$

↳ find the optimal subset $< W$

Knapsack problem: n pairs $(w_1, v_1)(w_2, v_2) \dots (w_n, v_n)$

Budget $W$

↳ find the optimal subset $< W$

↳ $v(S)$

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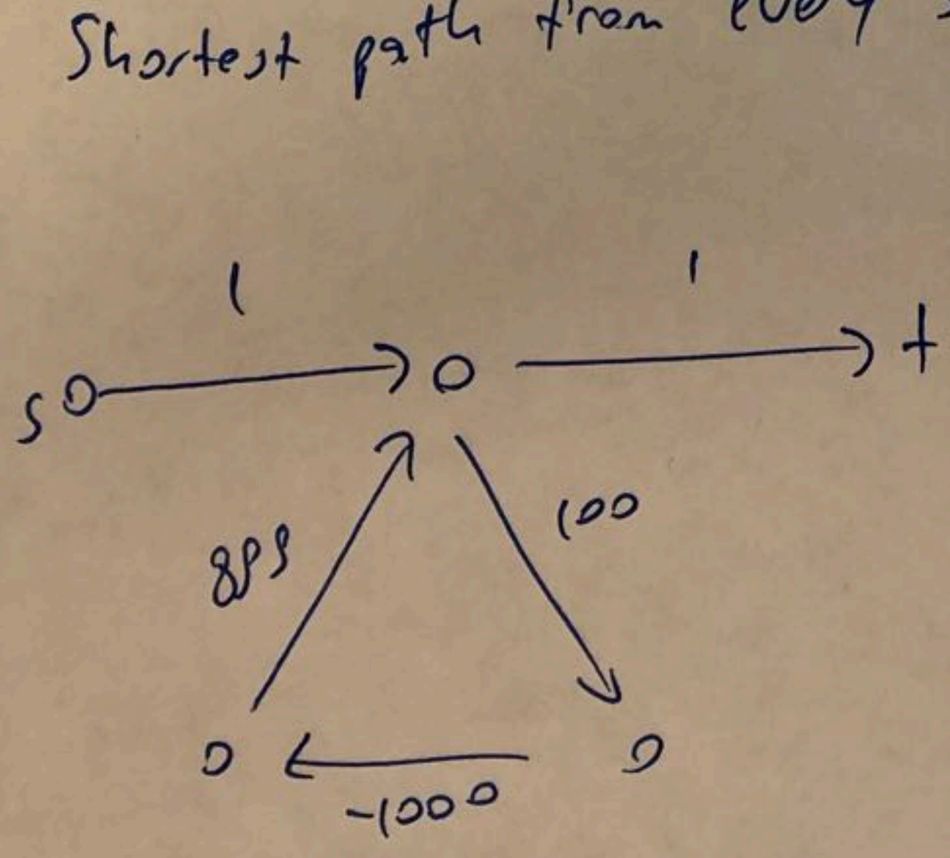
Shortest path problem

Input: ① Directed Graph $G=(V,E)$ ② vertex $t$

each edge has a cost $c_e$ $(c_e < 0)$

Output: Shortest path from every $s$ to $t$




Assume $G$ has no negative cycle

# Shortest Path Problem

**Input:** (•) Directed graph $G=(V,E)$, $\forall e \in E$ has cost $c_e$ ($c_e < 0$ is allowed)

(•) $t \in V$

**Output:** $\forall s \in V$, output a shortest $s$-$t$ path

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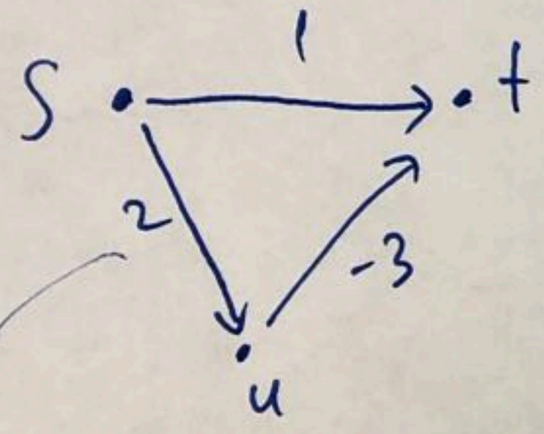
**Attempt 1:** Run Dijkstra for each $s \in V$




→ start Dijkstra on $s$, it'll pick $s,t$ as shortest path

BUT $s,u,t$ is the shortest $s$-$t$ path

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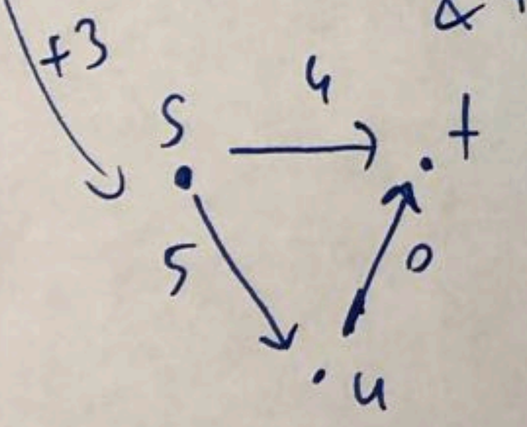
**Attempt 2:** Add a large enough positive number to each edge & then run Dijkstra on the new instance




→ In this graph, $s$-$t$ is shortest path

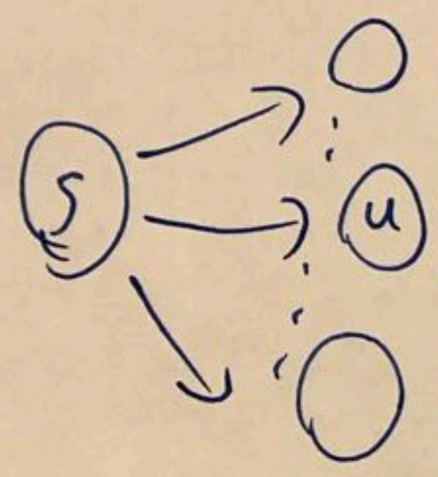
→ No known greedy / divide & conquer alg.

ASSUME: Only interested in cost of shortest s-t paths

Attempt 3: $OPT(s)$ = cost of a shortest s-t path $\forall s \in V$

(1) Poly many subproblem? ✓ $n$ subproblems

(2) Recurrence

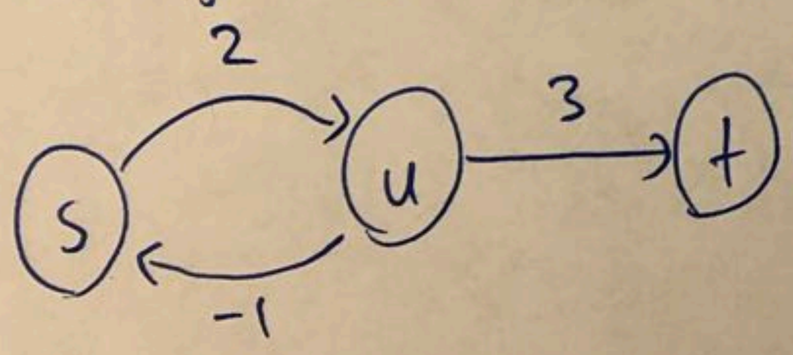



If a shortest s-t path starts with edge $(s,u)$ $s \neq t$

$$OPT(s) = c_{s,u} + OPT(u)$$

$$OPT(s) = \min_{\substack{w: \\ (s,w) \in E}} \{ c_{s,w} + OPT(w) \}$$

(3) Ordering among subproblems?




$$OPT(s) = c_{s,u} + OPT(u)$$
$$= 2 + OPT(u)$$

$$OPT(u) = \min \{ c_{u,t} + OPT(t), \; c_{u,s} + OPT(s) \}$$
$$= \min \{ 3 + OPT(t), \; -1 + OPT(s) \}$$

Issue: $OPT(s)$ depends on $OPT(u)$, $OPT(u)$ depends on $OPT(s)$ → there is no hope for total ordering

$os = 2 + ou$

$ou = \min\{3, os - 1\}$

$ou = \min\{3, ou + 1\}$

$ou = 3$

Attempt 4: $OPT(s, E')$ → cost of shortest s-t path only using edges in E'

$E' \subseteq E$

Solution: Introduce an implicit parameter in sub-problems

② Recursion

(s) → (u)

(If a shortest s-t path uses (s,u))

$$OPT(s, E) = C_{s,u} + OPT(u, E \setminus \{s,u\})$$

More generally: $OPT(s, E) = \min_{\substack{w: \\ (s,w) \in E}} \{C_{s,w} + OPT(w, E \setminus \{s,w\})\}$

③ Ordering? increasing order of $|E'|$ ✓

① How many sub-problems? $n \cdot 2^m$ → exponential!