Apr 22

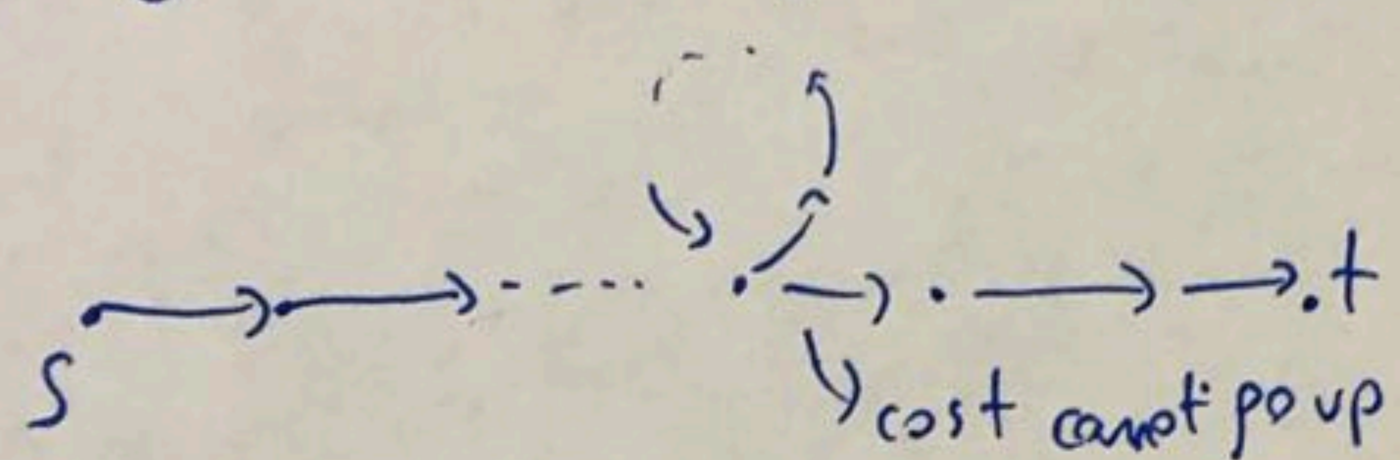
Attempt 5: Bellman-Ford Algo

$OPT(s, i)$ = cost of a shortest s-t path with $\leq i$ edges

$s \in V$, $i \geq 0$

<u>Prop</u>: If G has no negative cycle $\Rightarrow \forall s$ $\exists$ shortest s-t path that is <u>simple</u>

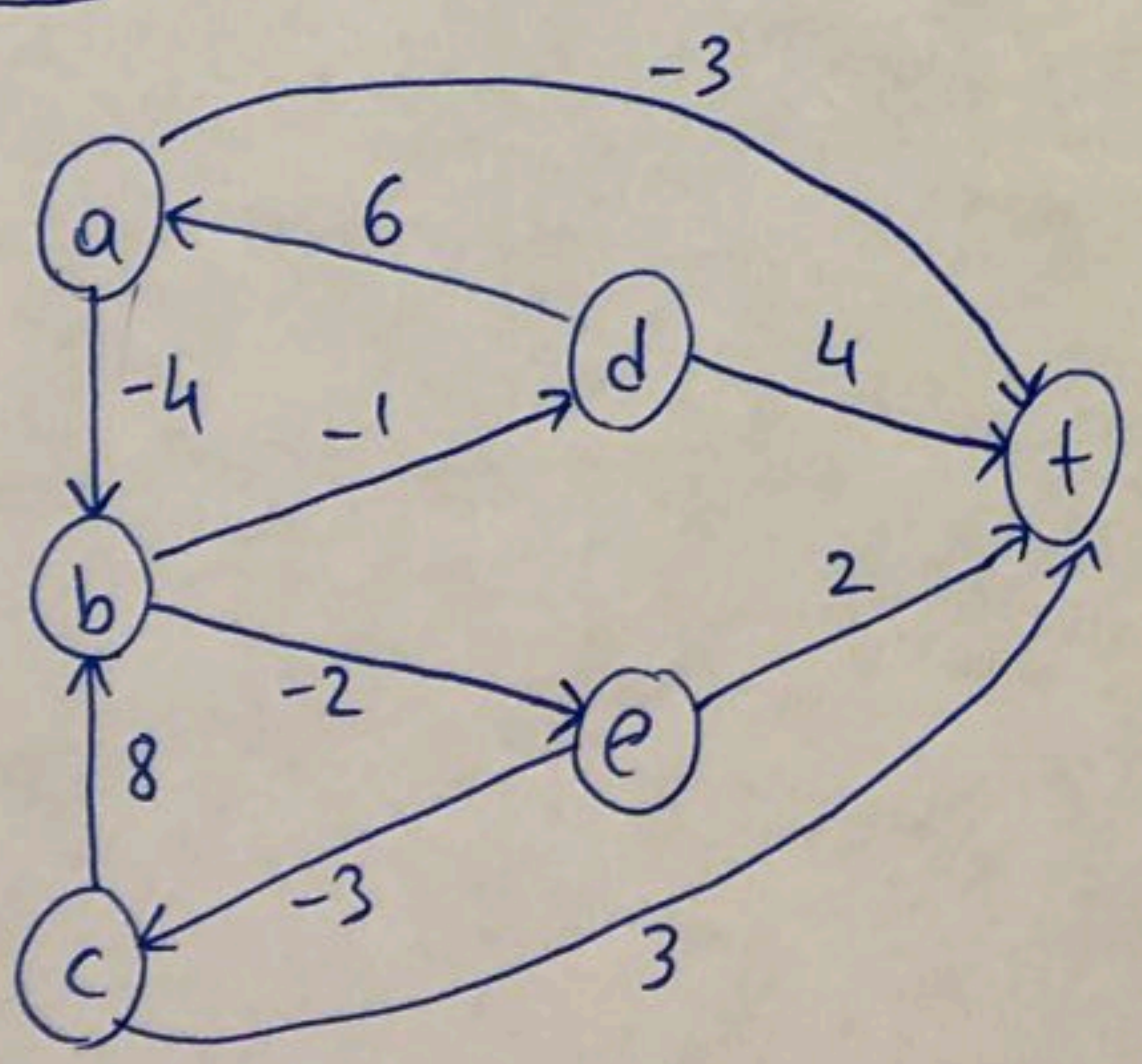
Pf(idea):



$OPT(S, i)$: $0 \leq i \leq n-1$

$\Rightarrow OPT(s, n-1)$ = cost of a shortest s-t path

<u>Goal</u>: Compute $OPT(s, n-1)$ $\forall s \in V$



Look at d

$OPT(d, 0) = \infty$ (as $d \neq t$)

$OPT(d, 1) = 4$ [d, t]

$OPT(d, 2) = \underset{(6-3)}{3}$ [d, a, t]

$OPT(d, 3) = 3$ [d, a, t]

$OPT(d, 4) = 6-4-2+2 = 2$ [d, a, b, e, t]

$OPT(d, 5) = 6-4-2-3+3 = 0$ [d, a, b, e, c, t]

$OPT(d, 6) = 0$

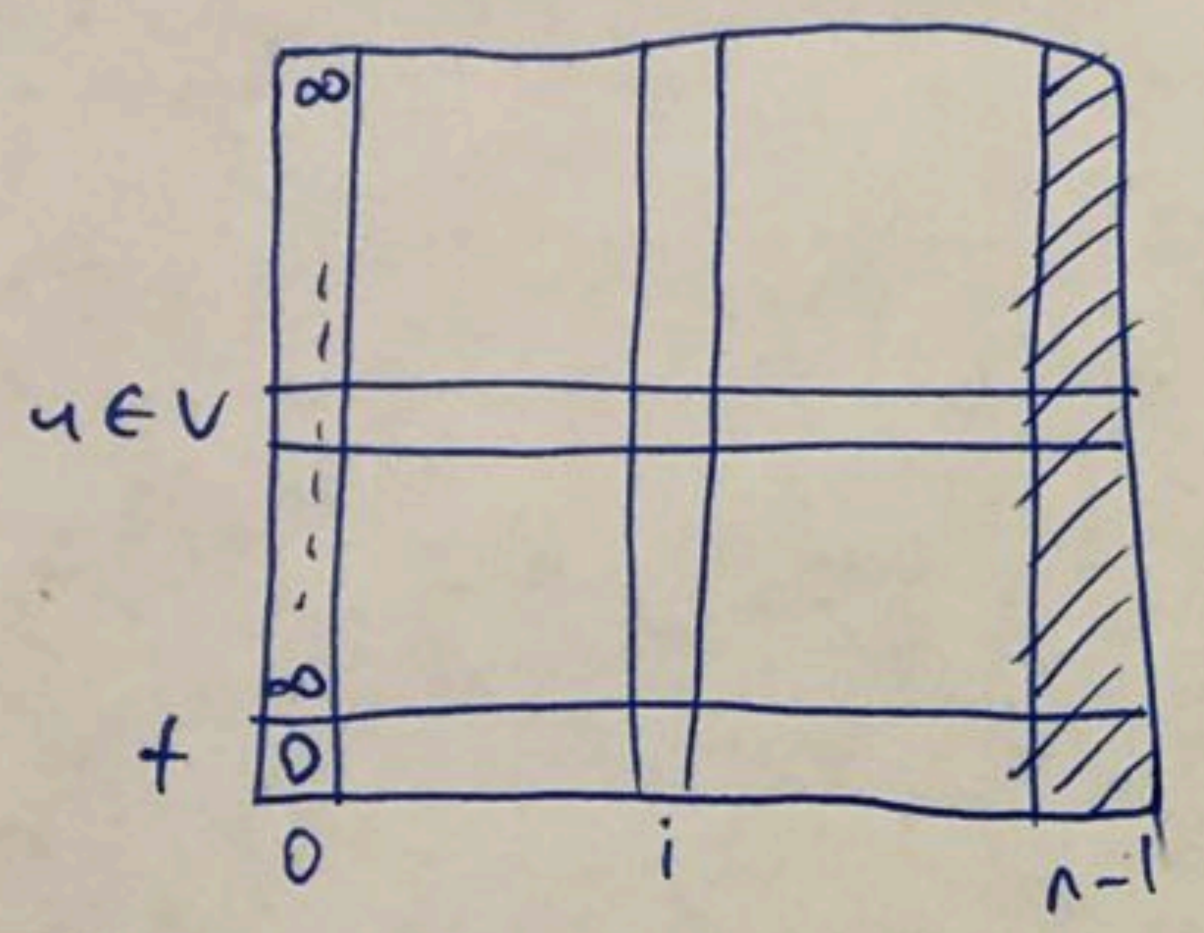
$OPT(d, 7) = 0$

$OPT(d, 6) = OPT(d, 5) = OPT(d, 7) \ldots$

By Prop.
$n = 6 \Rightarrow OPT(d, 5)$ is the cost of the shortest path

$OPT(s,i)$ = cost of a shortest s-t path with $\leq i$ edges

$s \in V$, $0 \leq i \leq n-1$



<u>Goal:</u> $M[u,i] = OPT(u,i)$

subproblems $= n^2$ poly many subproblems ✓

→ <u>Output</u>: $M[u,n-1]$ $\forall u \in V$

<u>Recurrence:</u> $OPT(t,0) = 0$

$OPT(u,0) = \infty$ $\forall u \neq t$

$OPT(u,i)$ for $i > 0$

<u>Case 1:</u> $\exists$ a shortest u-t path with $\leq i$ edges that actually uses $\leq i-1$ edges

$OPT(u,i) = OPT(u,i-1)$

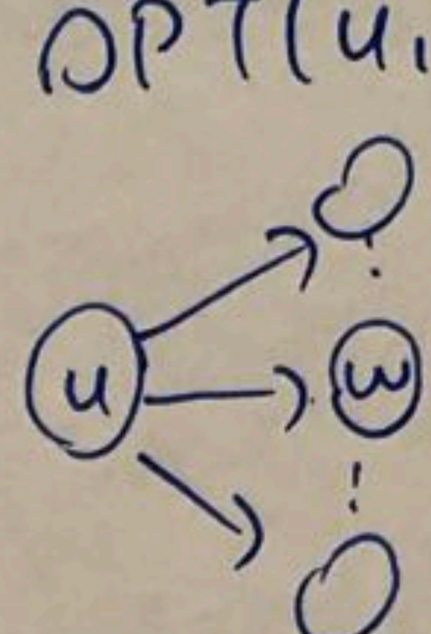
<u>Case 2:</u> All shortest u-t paths with $\leq i$ edges uses EXACTLY i edges

~~Know 1st edge is (u,w)~~ Know: 1^{st} edge is (u,w)

$$u \to \underbrace{w \cdots \to t}_{i-1 \text{ edges}}$$

$\Rightarrow$ shortest w-t path with $\leq i-1$ edges

$OPT(u,i) = c_{u,w} + OPT(w,i-1)$

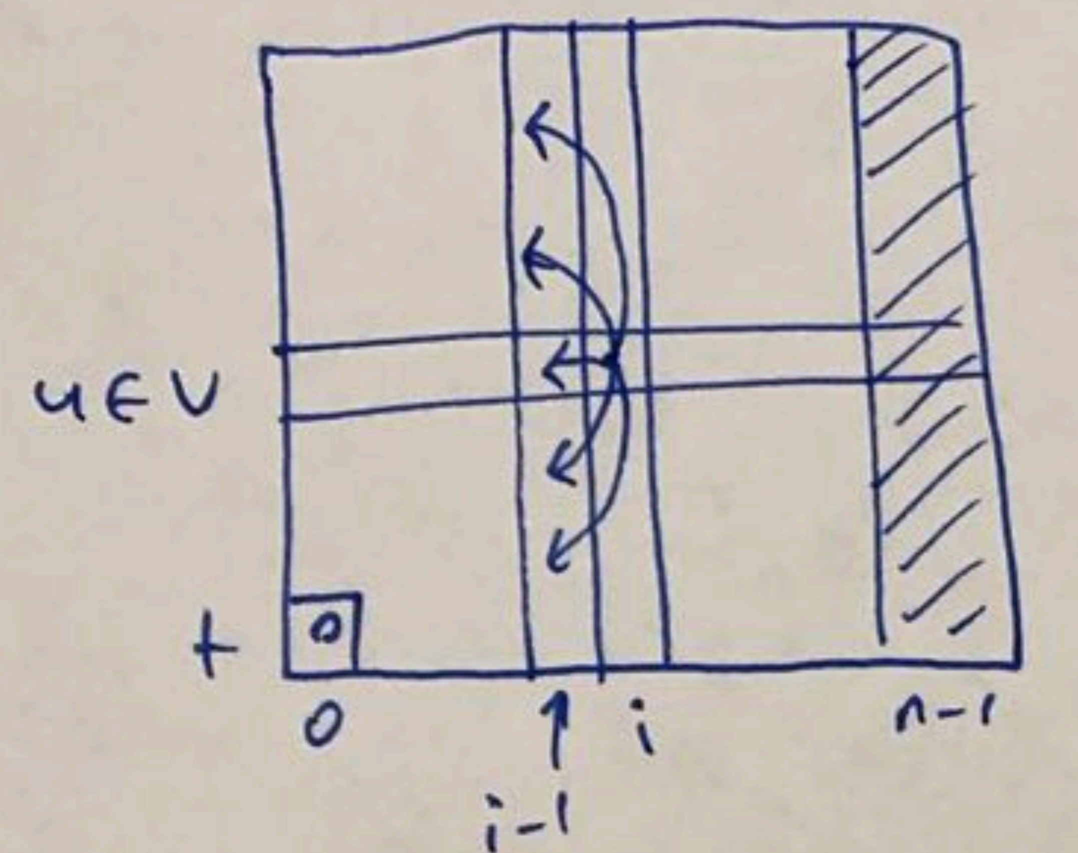


<u>Generally</u>, 1^{st} edge has to be from u to one of its neighbors

$$OPT(u,i) = \min_{w:\,(u,w)\in E} \{c_{u,w} + OPT(w,i-1)\}$$

OVERALL// $OPT(u,i) = \min\left\{OPT(u,i-1),\ \min_{w:\,(u,w)\in E}\{c_{u,w} + OPT(w,i-1)\}\right\}$ ✓

Ordering



$\Rightarrow$ i^{th} column only depends on column $i-1$

$\Rightarrow$ good ordering: column by column

(L to R $0, 1, \ldots, n-1$) ✓

Bellman-Ford Algo:

0. Allocate an $n \times n$ matrix M

1. $M[t,0] = 0$, $M[u,0] = \infty \ \forall u \neq t$

2. for $i = 1 \ldots n-1$

   for $u \in V$

   $$M[u,i] = \min\left\{ M[u,i-1], \min_{w:\,(u,w)\in E} \{ c_{u,w} + M[w,i-1] \} \right\}$$

3. Return $M[s, n-1] \ \forall s \in V$

	0	1	2	3	4	5
a	∞	-3				
b	∞					
c	∞					
d	∞					
e	∞					
t	0					

$$M[a,1] = \min\{ M[a,0], \min\{ -4 + M[b,0], -3 + M[t,0] \} \}$$

$$= \min\{ \infty, \min\{ -4 + \infty, -3 + 0 \} \}$$

$$= -3$$