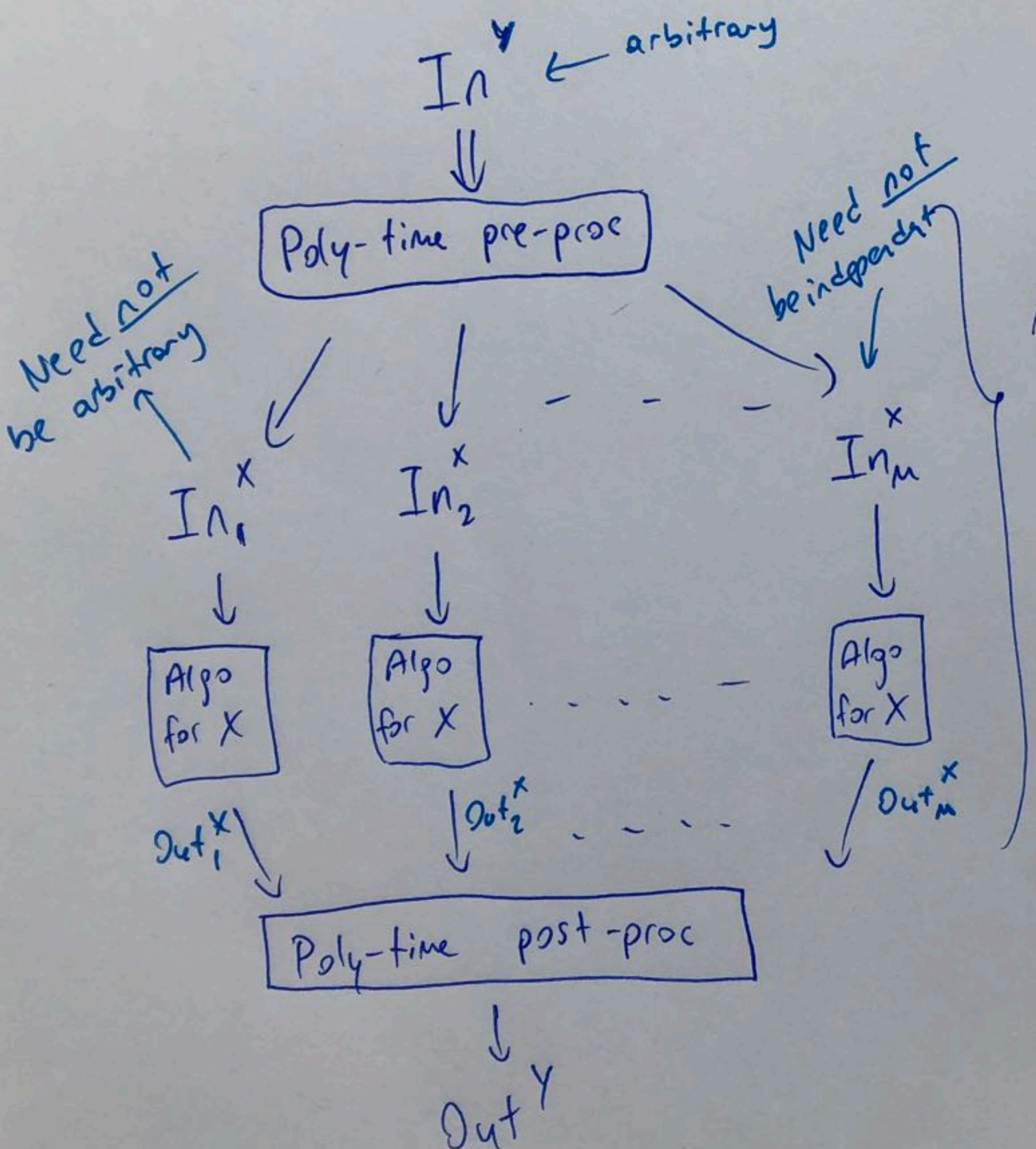
Apr 27

$$Y \leq_p X$$

$\rightarrow Y$ is poly time reducible <u>to</u> X

$\equiv$ poly time reduction <u>from</u> Y <u>to</u> X

Solve In^Y - - - - - - -> Out^Y



size of the input

$M = poly(N)$

$M = 1$ so far

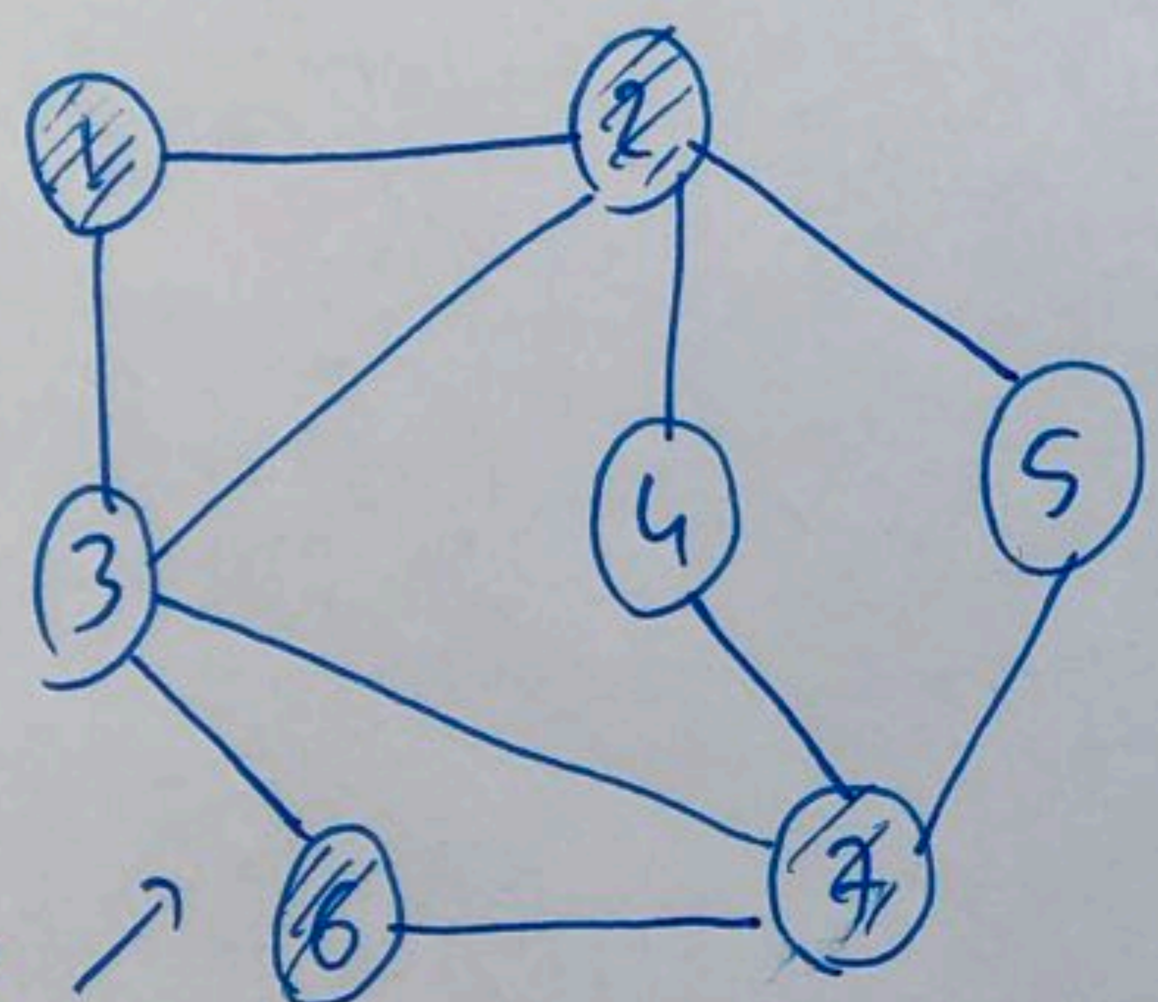
<u>Ex:</u> HW 2 Q2 $\leq_p$ Stable Matching

$(M = 1)$

Going forward: ONLY consider problems w/ Boolean output

Problem 1: Independent Set $G = (V, E)$

$S \subseteq V$ is an independent set if NO edges exist between nodes in S



G_0

Eg: $\{1,4\}$ ✓ $\{6\}$ ✓

$\{3,7\}$ X $\{1,4,5,6\}$ ✓

$\{1,4,7\}$ X

$\{3,4,5\}$ ✓

Problem:

Input: $G = (V, E)$; $0 \leq k \leq n$

Output: True iff $\exists$ an IS of size $\geq k$

Ex: $G_0, 2$: True $G_0, 4$: ✓ $G_0, 5$: X

Problem 2: Vertex Cover $G(V, E)$; $C \subseteq V$ is a vertex cover iff ALL edges in E have ≥ 1 end-point in C.

Ex: G_0 : $\{1,2,3,4,5,6,7\}$ ✓

$\{1,2,3,4,5,6\}$ ✓ $\Rightarrow$ Any subset of size $n-1$ is a VC

$\{1,2,6,7\}$ ✓

$\{2,3,7\}$ ✓ $\{1,7\}$ X

Problem: Input: $G = (V, E)$ $\quad 0 \leq k \leq n$

Output: True iff $\exists$ a VC of size $\leq k$

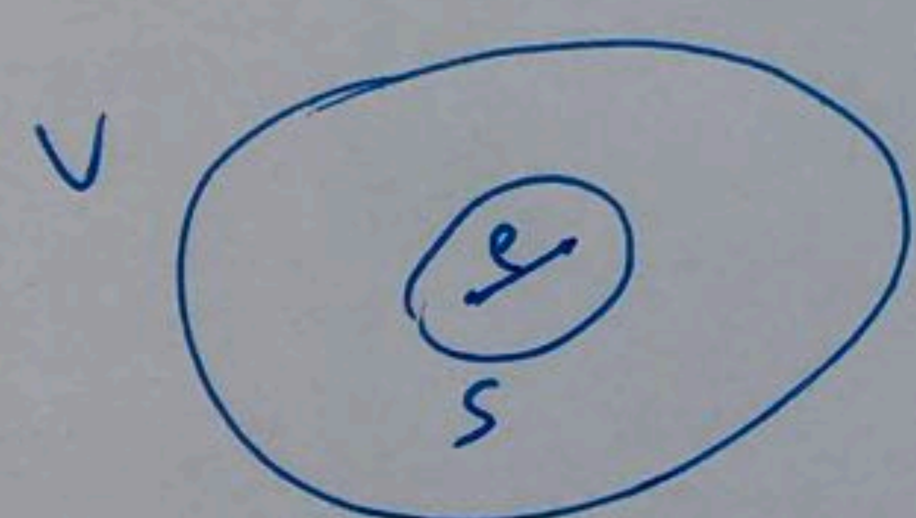
Ex: $G_0; 6$ ✓ $\quad G_0; 3$ ✓ $\quad G_0; 2$ ✗

THM: (1) $IS \leq_p VC$

(2) $VC \leq_p IS$

Lemma: $G = (V, E) \Rightarrow S \subseteq V$ is an IS iff $V \setminus S$ is a VC

Pf: $\Rightarrow$ Let S be an IS. Assume $V \setminus S$ is **NOT** a vertex cover

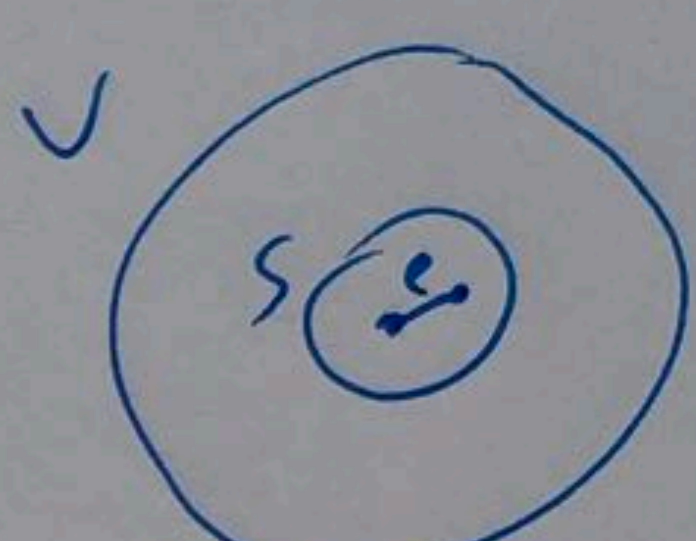


$\Rightarrow \exists$ an edge e with no endpoint in $V \setminus S$

$\Rightarrow$ Both endpoints of e are in S

$\Rightarrow S$ is not an IS $\rightarrow$ ✗

$\Leftarrow$ Let $V \setminus S$ be a vertex cover. But S is **not** an IS



$\Rightarrow \exists$ an edge "inside" S

$\Rightarrow V \setminus S$ is not a VC $\rightarrow$ ✗ $\blacksquare$

COR: G has an IS of size $\geq k \Leftrightarrow G$ has a VC of size $\leq n-k$

$\rightarrow IS \leq_p VC$

Pf: Input: (G,k) for IS

$\Rightarrow (G, n-k)$ for VC

$$(G,k) \xrightarrow{G(n-k)} \boxed{\text{Algo for VC}} \xrightarrow{T/F}$$

$\Rightarrow$ Similarly can show $VC \leq_p IS$

Satisfiability (SAT) problem — "the" NP problem in "practice"

Variables: $X = \{X_1, \dots X_n\}$ (each $X_i \in \{0,1\} \equiv \{F, T\}$)

Term\Literal: X_i, $\overline{X_i}$

Clause: OR or disjunction of literals

$t_1 \vee t_2 \vee t_3 \dots \vee t_\ell$ eg $n=3$ $X_1 \vee \overline{X_2}$

A SAT formula: Conjunction (AND) of clauses

$C_1, C_2 \dots, C_m \equiv C_1 \wedge C_2 \wedge C_3 \dots \wedge C_m$

Ex. $(X_1 \vee \overline{X_2}), (\overline{X_1} \vee \overline{X_3}), (X_2 \vee \overline{X_3})$

An assignment: $U: X \to \{0,1\}$

Ex: $X_1 = 0$, $X_2 = 0$, $X_3 = 0$ $(0,0,0)$ $(1,1,1)$

An assignment satisfies a clause C if C evaluates to T with the assignment

Ex. $X_1 \vee \overline{X_2}$ for $(0,0,0)$

$= 0 \vee \overline{0} = 0 \vee 1 = 1$