Feb 10

THEOREM: For any input (M, W, $2n$ preference lists) the GS algorithm outputs a stable matching.

$\Rightarrow$ every input has a stable matching.

LEMMA 1: For every input, the GS algo. terminates in $\leq n^2$ iterations

LEMMA 2: The output of GS algo (S) is a perfect matching

LEMMA 3: S has no instability.

Lemmas 1+2+3 $\Rightarrow$ Theorem

Pf idea Lemma 1: In each iteration, a new proposal is made (from w to m)

$\Rightarrow$ # iterations = # proposals $\leq$ # pairs $(w,m) = |W \times M|$

$= |W| \cdot |M| = n \cdot n = n^2$

(Pf details are on pg 7 in book)

Obs 0: S is a matching.

Obs 1: Once a man gets engaged, he keeps getting engaged to better women

Obs 2: If w proposes to m after m' $\Rightarrow$ m' > m in L_w

LEMMA 4: If at the end an iteration, w is free $\Rightarrow$ w has NOT proposed to all men.

Pf of Lemma 2: (Pf idea) Proof by contradiction (use Obs 0, Lemmas 1+4, algo. def)

(Pf details): Assume S is not a perfect matching.

$\Rightarrow$ (Obs 0, Algo def) $\exists$ a free woman w $\Rightarrow$ (Lemma 4) $\exists$ a man m that w has not proposed to. (★)

By Lemma 1, algo has terminated $\Rightarrow$ (Algo defn) All free woman have proposed to all men $\Rightarrow$ contradicts (★)