Mar 8

Greedy algorithm

0. $R = [n]$
1. $S = \emptyset$
2. While $R \neq \emptyset$
   - 2.1. Let i be the smallest index in R
   - 2.2. Add i to S
   - 2.3. Remove i from R
   - 2.4. Delete all j from R that conflicts i
3. Return $S^* = S$

Thm 1: S^* is an optimal solution.

↳ $\forall$ inputs, among all possible valid schedules for given input, S^* has the maximum number of jobs.

Ex. 1. Algo terminates

Ex. 2. S^* is a valid schedule

Pf. of correctness of greedy alg → Greedy stays ahead
→ Exchange argument (min-max lateness in Sec. 4.2)

Let $\mathcal{O}$ be an optimal solution

Ex. 3 Convince yourself that such an $\mathcal{O}$ exists

Idea: $S^* = \mathcal{O}$ $\longmapsto \mathcal{O}$
$\longmapsto S^*$

problem: can have >1 optimal solutions.

Idea: $|S^*| = |\mathcal{O}|$

Notation: $S^* = \{i_1, \ldots, i_k\}$ $f(i_1) \le f(i_2) \le \ldots \le f(i_k)$

$\mathcal{O} = \{j_1, \ldots, j_m\}$ $f(j_1) \le f(j_2) \ldots \le f(j_m)$

$\boxed{\text{Thm 1': } k = m}$

Claim 1: $k \le m$ (as $\mathcal{O}$ is optimal)

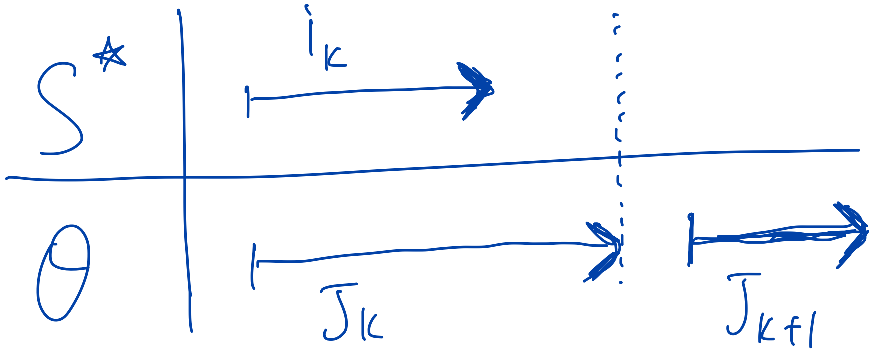
Lemma 1: "Greedy stays ahead"
$\forall\, 1 \le \ell \le k$ $f(i_\ell) \le f(j_\ell)$

(Assume Lemma 1 is true)

Pf. idea Thm 1': By contradiction: $k \ne m \underset{\text{By claim 1}}{\Rightarrow} k < m$

$\Updownarrow$

$m \ge k+1$



By Lemma 1,

$$f(i_k) \le f(j_k)$$

→ Consider the situation after i_k is added to S

→ $j_{k+1} \in R$ (as j_{k+1} does not conflict with any i_ℓ, $\ell \le k$)

→ R is non-empty

⟹ Greedy alg did not terminate.

⟹ contradiction.

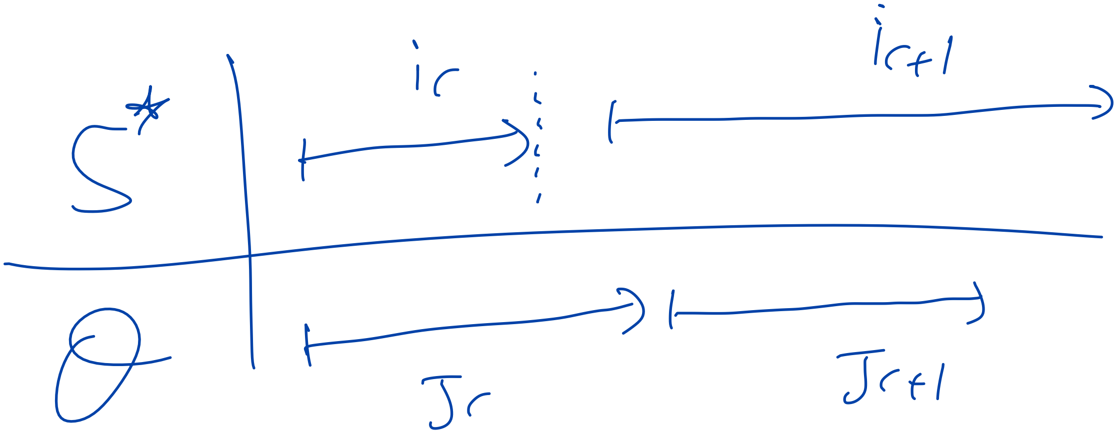
Pf. idea of Lemma 1: By induction on ℓ

Base case: $\ell = 1$ $\quad f(i_1) \le f(j_1)$ → By alg. defn $f(i_1)$ is the smallest finish time

I.H. $f(i_\ell) \le f(j_\ell)$ $\forall\ 1 \le \ell \le r$ $(r \ge 1)$

I.S. Show $f(i_{r+1}) \le f(j_{r+1})$

For the sake of contradiction, $f(i_{r+1}) > f(j_{r+1})$



(.) consider the situation after i_r is added to S by alg.

$\Rightarrow i_{r+1} \in R \Rightarrow i_{r+1}$ cannot be picked by greedy (as $f(j_{r+1}) < f(i_{r+1})$)

$\Rightarrow j_{r+1} \in R$

$\Downarrow$

contradiction $\blacksquare$