Mar 24

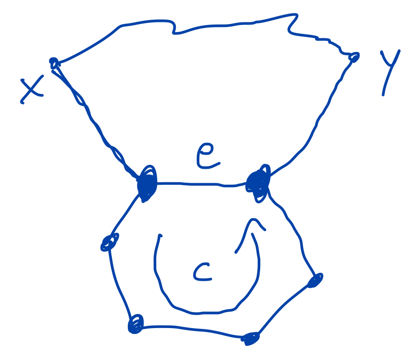
Prop. Let $c_e > 0 \ \forall e \in E$. Then the optimal solution T is a tree.
$T = (V, E')$

Pf. idea By contradiction.

Assume T is not a tree.

$\Longrightarrow$ $\exists$ a cycle in T.

as T is connected



Fix any cycle c in T.

Fix any edge e in c.

Delete e from T to get $T' = (V, E' \setminus \{e\})$

Claim 1: T' is connected ; **Claim 2:** $c(T') < c(T)$

$\Longrightarrow$ Claim 1+2 $\Rightarrow$ contradiction (optimality of T)

Pf. of Claim 2: $c(T') = c(T) - c_e < c(T)$ as $c_e > 0$

Pf. of Claim 1: Consider $x, y \in V$

Case 1: $\exists$ an x-y path that does not use e

Case 2: All x-y path use edge e $\Longrightarrow$ use the rest of c instead of e

$\Downarrow$

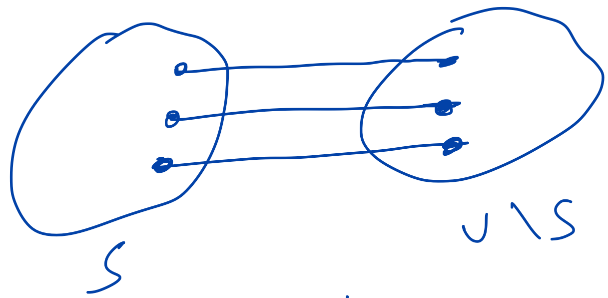
(x, y) is connected in T'

Cut Property Lemma

$S \neq V$

Assume: All c_e's are distinct.

For all cuts $(S, V \setminus S)$ s.t. $S \neq \emptyset$, $V \setminus S \neq \emptyset$



Consider all cutting edges

Let e be the cheapest crossing edge

$\Longrightarrow$ e is in **ALL** MSTs

Assume cut-property lemma is true (c_e's are distinct)

THM 1: Prim's alg. is correct

Pf. idea: Consider the run of the alg. when it's about to add e to T.

Goal: e is the cheapest crossing edge for some cut $(S, V \setminus S)$ $\xRightarrow{\text{cut prop. lemma}}$ this is a safe choice

Apply the cut property lemma on $(S, V \setminus S)$ where S is from Prim's alg.

Claim 1: $S \neq \emptyset$ $(u \in S, s \in S)$

Claim 2: $S \neq V$ $(w \notin S)$

Claim 3: e is the cheapest crossing edge (follows from alg. statement)

$\Longrightarrow$ every edge added by Prim is correct/safe

Claim 4: At the end of each iteration (S, T) is connected

$\Longrightarrow$ at the end of the alg. (V, T) is connected

$\boxed{\text{Pf: Ex}}$

Claims $1+2+3+4 \Rightarrow$ Thm 1 $\blacksquare$

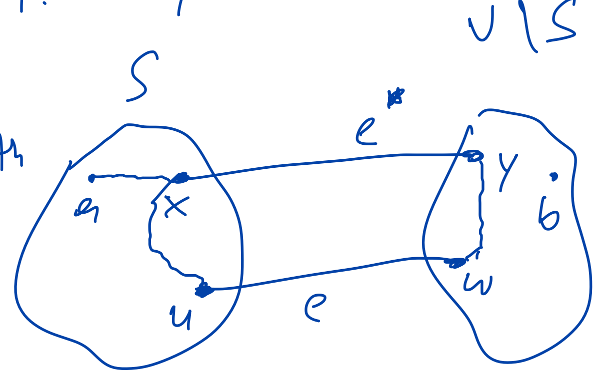
Proof of Cut-property lemma

(Pf. idea) By contradiction.

Assume not $\Longrightarrow$ $\exists$ a cut $(S, V \setminus S)$ s.t. e is the cheapest crossing edge

$\exists$ an MST T s.t. $e \notin T$

Since T is connected $\Longrightarrow$ $\exists$ u,w path in T



$u \in S, w \notin S \Longrightarrow \exists\, x \in S, y \notin S$ s.t. $(x,y) = e^*$ is an edge in T.

Define: $T' = (T \setminus \{e^*\}) \cup \{e\}$ as $c_{e^*} > c_e$

Claim 1: $c(T') = c(T) - c_{e^*} + c_e < c(T)$

Claim 2: T' is connected

Case 1: a-b path doesn't use e^* $\Rightarrow$ ✓

Case 2: a-b path does use e^* $\Rightarrow$ take "scenic route" ✓

Claims 1 + 2 $\Rightarrow$ T is NOT an MST $\Rightarrow$ contradiction ∎