Apr 12

Weighted Interval Scheduling

Simplified problem: Only compute the value of an optimal solution

$OPT(j)$ = value of an optimal solution for $[j]$ $\quad 0 \le j \le n$

$\{(s_1, f_1, v_1), (s_2, f_2, v_2) \ldots (s_n, f_n, v_n)\}$

$f_1 \le f_2 \le f_3 \ldots \le f_n$

Goal: Compute $OPT(n)$

Def: O_j be an optimal soln for $[j]$

$OPT(j) = v(O_j)$

$OPT(0) = 0$

Goal: $OPT(j) = \max\{OPT(j-1), v_j + OPT(p(j))\}$ $\quad p(j) < j$ TBA

Case 1: $j \notin O_j$

Claim 1: O_j is also optimal for $[j-1]$

$$\Longrightarrow OPT(j) \underset{\substack{\text{by def}^n \\ \text{of } O_j}}{=} v(O_j) \underset{\substack{\text{by claim} \\ 1}}{=} OPT(j-1)$$

Pf (idea) of Claim 1: Assume O_j is not optimal for $[j-1]$

$\exists$ a schedule O' for $[j-1]$ s.t. $v(O') > v(O_j)$

But $\mathcal{O}'$ is also valid for $[J] \Longrightarrow$ contradicts the assumption that $\mathcal{O}_J$ is optimal for $[J]$

Case 2: $J \in \mathcal{O}_J$

Def: $p(J)$: the largest $i < J$ s.t. i & J do not conflict

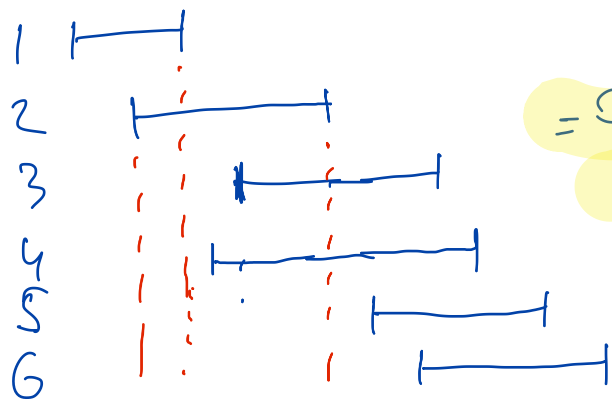
$= 0$ if no such $i\ \exists$

(i) $p(J)+1, p(J)+2, \ldots, J-1$ are in conflict with J

(ii) $1, 2, \ldots, p(J)$ will NOT be in conflict with J

Ex.

$p(1) = 0$
$p(2) = 0$
$p(3) = 1$
$p(4) = 1$
$p(5) = 2$
$p(6) = 2$



Claim 2: $\mathcal{O}_J \setminus \{J\}$ is an optimal solution for $[p(J)]$.

$$\Longrightarrow OPT(J) = v(\mathcal{O}_J) = v_J + v(\mathcal{O}_J \setminus \{J\}) \underset{\text{by Claim 2}}{=} v_J + OPT(p(J))$$

Pf(idea) of Claim 2: Assume $\mathcal{O}_J / \{J\}$ is NOT optimal for $[p(J)]$

$\Rightarrow \exists\ \theta'$ which is a valid schedule for $[p(j)]$

& $v(\theta') > v(\theta_j \setminus \{j\})$

<u>Note:</u> $\theta' \cup \{j\}$ is a valid schedule for $[j]$

but $v(\theta' \cup \{j\}) = v(\theta') + v_j > \underbrace{v(\theta_j \setminus \{j\}) + v_j}_{v(\theta_j)}$

$\Rightarrow$ contradicts with the optimality of θ_j for $[j]$

<u>Ex.</u> Compute $p(j)$ for $1 \leq j \leq n$ in $O(n.\log n)$ time

<u>Bonus</u>. $\Omega(n.\log n)$ comparisons to compute $p(j)$ for all j