Apr 16

SUBSET SUM PROBLEM

Input: ① n integers $w_1, w_2, \ldots, w_n$; $w_i > 0$

② Budget $W \geq 0$

Output: A subset $S \subseteq [n]$ s.t.

(i) $\sum_{i \in S} w_i \leq W$

(ii) $\max W(S) = \sum_{i \in S} w_i$

Ex. $n = 3$ $\quad w_A = 1 \quad w_B = 3 \quad w_C = 3$

(i) $W = 7 \Rightarrow$ opt soln $\{A, B, C\}$

(ii) $W = 6 \Rightarrow$ opt soln $\{B, C\}$

(iii) $W = 5 \Rightarrow$ opt solns $\{A, B\}$ $\{A, C\}$

Simpler Q: $\max |S|$ (instead of $W(S)$)

Greedy alg: Sort in increasing order of w_i & pick as many as you can without exceeding the budget

Ex. prove this is optimal (Greedy stays ahead)

Original problem: maximize $W(S)$

Try: Greedy alg. counterexample → $w_1 = 10$, $w_2 = 30$, $w_3 = 30$ $\qquad W = 60$

Greedy picks $\{1,2\}$ but optimal soln is $\{2,3\}$

Note: No known greedy alg.

Dynamic Program for Subset Sum problem

Goal: Compute $W(S) = \sum_{j \in S} w_j$ for an optimal S

O_j: be an optimal solution for $1, \ldots, j$ (don't have to be sorted)

$OPT(j) = w(O_j)$

Case 1: $j \notin O_j$ $\qquad OPT(j) = OPT(j-1)$

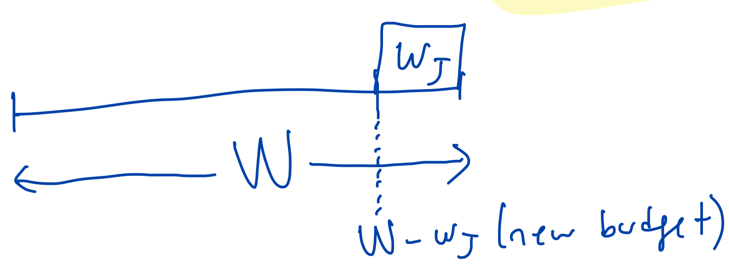
Claim: O_j is also optimal for $w_1, w_2, \ldots, w_{j-1}$

Case 2: $j \in O_j$

Q: What can we say about $O_j \setminus \{j\}$?

Hope: $O_j \setminus \{j\}$ is also optimal for $w_1, w_2, \ldots, w_{j'}$ for some $j' < j$

If so, $OPT(j) = w_j + OPT(j')$



(assume $w_j \leq W$)

Solution: Keep track of budget and J

$OPT(B,J)$ = weight of an optimal solution for $w_1, w_2, \ldots, w_J$ AND budget B

Assume $J \in \text{optimal}(w_1,\ldots,w_J; B)$ Assume $w_J \le B$

$$\Rightarrow OPT(B,J) = w_J + OPT(B-w_J, J-1) \quad (1)$$

$J \notin \text{optimal}(w_1,\ldots,w_J; B)$

$$\Rightarrow OPT(B,J) = OPT(B, J-1) \quad (2)$$

$$w_J > B \Rightarrow OPT(B,J) = OPT(B,J-1)$$

Overall recursion: If $w_J > B \Rightarrow OPT(B,J) = OPT(B,J-1)$

else $OPT(B,J) = \max\{w_J + OPT(B-w_J, J-1), OPT(B,J-1)\}$

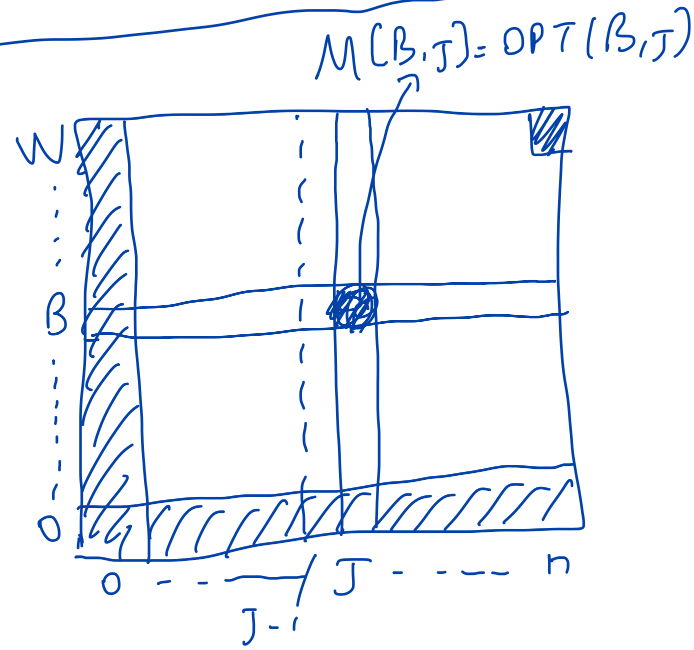
Q1: What entry of M is our final output $(w_1, w_2, \ldots w_n; W)$?

A1: $M[W,n] = OPT(W,n)$

Q2: Initial values?

A2: $M[B,0] = 0 \;\; \forall\; 0 \le B \le W$

$M[0,J] = 0 \;\; \forall\; 0 \le J \le n$



Q3: How many subproblems do we have?

A3: $(W+1).(n+1) \rightarrow$ poly(n) If W is poly(n)

Q4: Recurrence?

A4: Done

Q5: Ordering among subproblems?

A5: Go column by column as knowing $(j-1)^{th}$ column is enough to compute j^{th} column