Apr 19

Subset Sum $(w_1, \dots, w_n; W)$

$\left.\begin{array}{l} \text{0. Allocate a } (W+1)(n+1) \text{ matrix } M \leftarrow O((n+1)(W+1)) \\ \text{1. } M[B,0] = 0 \quad \forall\ 0 \le B \le W \leftarrow O(W) \\ \text{2. for } J = 1 \dots n \\ \qquad \text{for } B = 0 \dots W \\ \qquad\quad \left.\begin{array}{l} \text{if } (w_J > B) \\ \qquad M[B,J] = M[B,J-1] \\ \text{else} \\ \qquad M[B,J] = \max\{w_J + M[B-w_J, J-1],\ M[B,J-1]\} \end{array}\right\} O(1) \\ \text{3. Return } M[W,n] \end{array}\right\} O(n.W)$

Obs. $O(W)$ space if we're only interested in $OPT(W,n)$

But need $\Omega(n.W)$ space if we want an actual subset

$n=3 \quad w_1 = 1 \quad w_2 = 2 \quad w_3 = 2, \quad W = 3$

$B \downarrow$ / $J \rightarrow$	0	1	2	3
3	0	1	3.	//////
2	0	1	2	
1	0	1	1	
0	0	0	0	0

$M[1,1] = \max\{1 + M[1-1,0],\ M[1,0]\}$

$= \max\{1+0, 0\} = 1$

$M[2,1] = \max\{1 + M[2-1,0], M[2,0]\}$

$= \max\{1+0,\ 0\} = 1$

$\vdots$

$M[3,3] = \max\{2 + M[3-2,2], M[3,2]\}$

$= \max\{2+1, 3\} = 3$

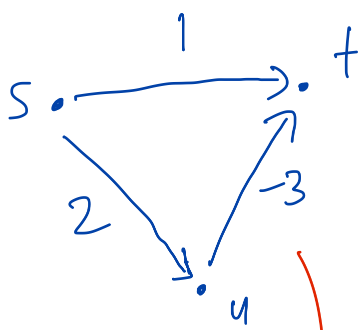
Shortest Path Problem

Input: (•) Directed graph $G(V,E)$, $\forall\, e \in E$ has cost c_e ($c_e < 0$ is allowed)

(•) $t \in V$

Output: $\forall\, s \in V$, output a shortest s-t path

Attempt 1: Run Dijkstra for each $s \in V$

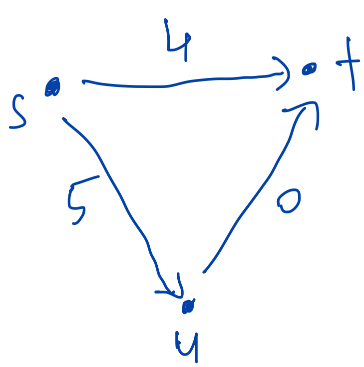


→ start Dijkstra on s, it'll pick s-t as the shortest path

BUT s,u,t is the shortest s-t path

+3

Attempt 2: Add a large enough positive number to each edge & then run Dijkstra on the new instance



→ In this graph, s-t is shortest path

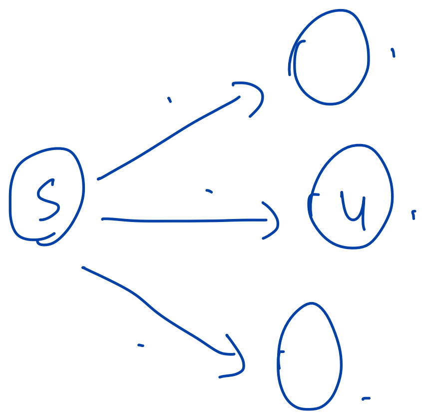
→ No known greedy / divide & conquer algo.

ASSUME: Only interested in cost of shortest s-t paths.

Attempt 3: $OPT(s)$ = cost of a shortest s-t path $\forall s \in V$

1. Poly many subproblems ✓ n subproblems

2. Recurrence ✓

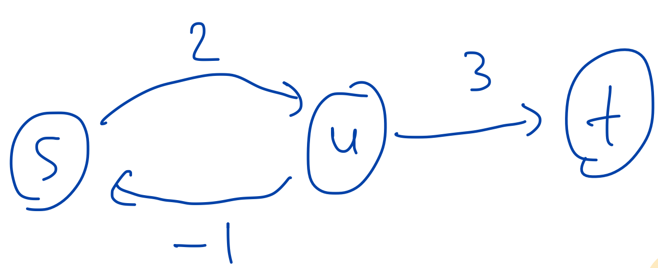


If a shortest s-t path starts with edge (s,u) $s \neq t$

$$OPT(s) = c_{s,u} + OPT(u)$$

$$OPT(s) = \min_{(s,w) \in E} \{ c_{s,w} + OPT(w) \}$$

3. Ordering among subproblems



$$OPT(s) = c_{s,u} + OPT(u)$$
$$= 2 + OPT(u)$$

$$OPT(u) = \min \{ c_{u,t} + OPT(t), c_{u,s} + OPT(s) \}$$
$$= \min \{ 3 + OPT(t), -1 + OPT(s) \}$$

Issue: $OPT(s)$ depends on $OPT(u)$
$OPT(u)$ depends on $OPT(s)$

There is no hope for total ordering